TWO-GRID ITERATION METHODS FOR LINEAR INTEGRAL EQUATIONS OF THE SECOND KIND ON PIECEWISE SMOOTH SURFACES IN \(\mathbb{R}^3\)

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Abstract. The numerical solution of integral equations of the second kind on surfaces in \(\mathbb{R}^3\) often leads to large linear systems that must be solved by iteration. An especially important class of such equations is boundary integral equation (BIE) reformulations of elliptic partial differential equations; and, in this paper BIEs of the second kind are considered for Laplace’s equation. The numerical methods used are based on piecewise polynomial isoparametric interpolation over the surface and the surface is also approximated by such interpolation. Two-grid iteration methods are considered for (1) integral equations with a smooth kernel function, (2) BIEs over smooth surfaces, and (3) BIEs over piecewise smooth surfaces. In the last case, standard two-grid iteration does not perform well, and a modified two-grid iteration method is proposed and examined empirically.

Key words. boundary integral equations, iteration methods

AMS subject classifications. primary 65R20; secondary 65N99, 65F10, 35J05

1. Introduction. The numerical solution of boundary integral equations that are reformulations of partial differential equations in \(\mathbb{R}^3\) often leads to the solution of very large systems of linear equations. At present, these linear systems are usually solved by iteration and, traditionally, a geometric series (also called a Neumann series) has been the basis of the iteration scheme. Here we consider two-grid methods for the solution of integral equations of the second kind. Two-grid methods usually converge faster than the geometric series and, in practical terms, they are usually comparable in speed to multigrid methods, while being simpler to program.

As our test case for solving a BIE, we consider the exterior Neumann problem for Laplace’s equation:

\[
\Delta u(A) = 0, \quad A \in D_e,
\]

\[
\frac{\partial u(P)}{\partial n_P} = f(P), \quad P \in S,
\]

\[
u_A = O(|A|^{-1}), \quad |\nabla u(A)| = O(|A|^{-2}) \quad \text{as } |P| \to \infty.
\]

The region \(D_e = \mathbb{R}^3 \setminus \bar{D}\), with \(D\) an open, bounded, and simply connected region, and \(S\) is the boundary of \(D\). The unit normal at \(P \in S\), directed into \(D\), is denoted by \(n_P\). Using Green’s third identity,

\[
4\pi u(A) = \int_S f(Q) \frac{1}{|A - Q|} dS_Q - \int_S u(Q) \frac{\partial}{\partial n_Q} \left[ \frac{1}{|A - Q|} \right] dS_Q, \quad A \in D_e.
\]

To find \(u\) on \(S\), we solve the integral equation

\[
2\pi u(P) + \int_S u(Q) \frac{\partial}{\partial n_Q} \left[ \frac{1}{|P - Q|} \right] dS_Q + [2\pi - \Omega(P)] u(P)
\]

\[
= \int_S f(Q) \frac{1}{|P - Q|} dS_Q, \quad P \in S.
\]
Here, $\Omega(P)$ denotes the inner solid angle of $S$ at $P \in S$. We assume

$$0 < \Omega(P) < 4\pi,$$

which avoids surfaces with cusps.

Symbolically, we write (1.3) as

$$\lambda u(P) - \int_S K(P, Q)u(Q)dS_Q = g(P), \quad P \in S$$

Under suitable additional assumptions on $S$, the operator $K$ maps $C(S)$ into $C(S)$ and is bounded; see [36]. It can also be shown for such surfaces that (1.4) is uniquely solvable for all $f \in C(S)$, with $(2\pi + K)^{-1}$ a bounded operator; throughout this paper, we assume this is the case. In addition, if $S$ has a parametrization that is differentiable with derivatives that are Hölder continuous with exponent $\gamma$, some $\gamma > 0$, then $K$ is a compact operator; see [26], [18].

The integral equation (1.3) can be solved in a number of ways. If $S$ is a "smooth surface," then in [4], [5], [8], a Galerkin method was presented with approximants based on spherical harmonics. For such problems, this is quite efficient and we recommend that it be considered seriously. Here we consider collocation methods based on triangulating $S$ and approximating $u$ by piecewise quadratic isoparametric interpolation over the triangulation. Such methods were considered previously in [6], [7], [9], and [14]. In [27], Galerkin methods for BIEs of the first kind were considered in a similar context. In the engineering literature such collocation methods have been popular, but usually with only piecewise constant or piecewise linear approximations.

In §2, we review some results on our collocation method that will be needed in later sections. In §3, we give a two-grid iteration method for a general integral equation $(2\pi + K)u = f$, with $K$ compact and approximated by a numerical integration operator $K_n, n \geq 1$. For BIEs this is too expensive a method, and in §4 we present a variant that follows ideas of Hackbusch, [19], [20, Chap. 16]. Convergence results are given for $S$ smooth, and numerical examples are given. In §5, we extend the iteration method to $S$, a piecewise smooth surface, discussing the difficulties inherent in this case. The major difficulty traces back to the lack of compactness for $K$ when $S$ is only piecewise smooth; this requires a modified iteration method. This is analyzed for planar BIEs in Atkinson and Graham [13]; some of these ideas are generalized here to handle BIEs for $S$ a piecewise smooth surface.

2. Preliminaries. We consider the integral equation

$$\lambda u(P) - \int_S K(P, Q)u(Q)dS_Q = g(P), \quad P \in S$$

with the integral operator $K$ a compact operator from $C(S)$ into $C(S)$. It is assumed that the parameter $\lambda$ is nonzero and is not an eigenvalue of $K$; thus $(\lambda - K)^{-1}$ exists as a bounded operator from $C(S)$ onto $C(S)$.

Assume that $S$ can be decomposed as

$$S = S_1 \cup \ldots \cup S_J$$

with each $S_i$ the image of a polygonal region $R_i$ in the plane $\mathbb{R}^2$.

$$F_i : R_i \overset{1-1}{\text{onto}} S_i, \quad i = 1, \ldots, J.$$ 

We assume that $F_i \in C^6(R_i)$. In addition, the subsurfaces $S_i$ are to intersect along only common boundary points. We triangulate each $S_i$ by first triangulating $R_i$. Then this triangulation
is mapped onto $S_i$ using $F_i$. Let $\{\Delta_{i,k} \mid 1 \leq k \leq N_i\}$ be a triangulation of $R_i$ and define $\Delta_{i,k} = F_i(\hat{\Delta}_{i,k})$, $1 \leq k \leq N_i$, $1 \leq i \leq J$. Collect together the triangulations of each $S_i$, and refer to them collectively as

$$T_N = \{\Delta_1, \ldots, \Delta_N\}.$$ 

We assume a number of properties for this triangulation of $S$. When two sections $S_i$ and $S_j$ have a section $\Gamma$ of their boundary curves in common, then the triangulations $\{\Delta_{i,k}\}$ and $\{\Delta_{j,l}\}$ must also agree along this common curve $\Gamma$. In addition, if $S$ is only a piecewise smooth surface, then (1) the union of the edges of all $\Delta_k \in T_N$ must contain the union of the edges of $S$, and (2) the set of all vertices of the $\Delta_k \in T_N$ must contain all vertices of the original surface $S$.

For convenience in defining integration and interpolation over the triangulation, we introduce a standard parametrization of each triangle $\Delta_i \in T_N$. Define the unit simplex:

$$\sigma = \{(s, t) \mid 0 \leq s, t, s + t \leq 1\}.$$ 

For $\Delta_{i,k} \subset S_i$, define $m_{i,k} : \sigma \rightarrow \Delta_{i,k}$ by

$$m_{i,k}(s, t) = F_i(u^k_{i,1} + t \hat{v}_{k,2}^i + s \hat{v}_{k,3}^i), \quad (s, t) \in \sigma$$

with $u = 1 - s - t$ and $\{\hat{v}_{k,1}^i, \hat{v}_{k,2}^i, \hat{v}_{k,3}^i\}$ the vertices of $\hat{\Delta}_{i,k}$. When referring to the triangulation $\Delta_i \in T_N$, we use $m_i : \sigma \rightarrow \Delta_i$ and the vertices of $\hat{\Delta}_i, \Delta_i$ are denoted by $\{\hat{v}_{l,1}, \hat{v}_{l,2}, \hat{v}_{l,3}\}$ and $\{v_{l,1}, v_{l,2}, v_{l,3}\}$, respectively. It is also necessary to introduce some other node points in $\Delta_i$.

Let $q_1, \ldots, q_6 \in \sigma$ be defined by

$$q_1 = (0, 0), \quad q_2 = (0, 1), \quad q_3 = (1, 0),$$

$$q_4 = (0, .5), \quad q_5 = (.5, .5), \quad q_6 = (.5, 0).$$

For $m_i : \sigma \rightarrow \Delta_i$, let $v_{i,j} = m_i(q_j), j = 1, \ldots, 6$. The nodes $v_{1,4}, v_{1,5}, v_{1,6}$ are called the “midpoints” of the sides of $\Delta_i$, although they are usually not true midpoints and are used to define quadratic isoparametric interpolation over $\Delta_i$. Collectively, the nodes of the triangulation $T_N$ are also referred to by

$$V_N = \{v_1, \ldots, v_{N\sigma}\}.$$ 

For a surface $S$ that encloses a simple closed region,

$$N_v = 2(N + 1).$$

**Quadratic interpolation.** Given $f \in C(\Delta_i)$, for any $1 \leq l \leq N$, define a quadratic interpolant of $f(m_i(s, t))$ by

$$f(m_i(s, t)) \approx \sum_{i=1}^{6} f(v_{l,i}) \ell_i(s, t), \quad (s, t) \in \sigma$$

with $\ell_1, \ldots, \ell_6$ the quadratic basis polynomials for which

$$\ell_i(q_j) = \delta_{ij}.$$ 

For $f \in C(S)$, denote the piecewise quadratic isoparametric interpolation of $f$, defined by (2.6), by $\mathcal{P}_N f$. 


The operator $P_N$ is a bounded projection on $C(S)$, and its properties were discussed in [6]. In particular,

\[(2.7) \quad \|P_N\| = \frac{5}{3}\]

and for $f \in C^3(S_i), i = 1, \ldots, J$,

\[(2.8) \quad \|f - P_N f\|_\infty = O(h^3),\]

where

\[h = \text{Max } \text{diameter } (\Delta_i).\]

In dealing with integrals over $S$, we write

\[(2.9) \quad \int_S g(Q) dS_Q = \sum_{k=1}^{N} \int_{\Delta_k} g(Q) dS_Q\]

\[= \sum_{k=1}^{N} \int_{\sigma} g(m_k(s, t)) |D_s m_k \times D_t m_k| ds \, dt\]

with $D_s m_k \equiv (\partial m_k(s, t)/\partial s)$ and similarly for $D_t m_k$. From (2.4), we must have parametrizations of each $S_i$ that are explicitly differentiable. To avoid this requirement, we introduce an approximating surface $\tilde{S}$ to approximate $S$, and the surface $\tilde{S}$ is used to calculate approximations to the surface differentials in (2.9).

For each $\Delta_k \in T_N$, define an interpolating mapping $\tilde{m}_k$ by

\[(2.10) \quad \tilde{m}_k(s, t) = \sum_{i=1}^{6} m_k(q_i) \ell_i(s, t) = \sum_{i=1}^{6} v_{ki} \ell_i(s, t).\]

By the interpolation error results, and by using the assumed differentiability of $m_t$,

\[(2.11) \quad \text{Max } \text{Max } |m_k(s, t) - \tilde{m}_k(s, t)| = O(h^3).\]

Denote $\tilde{\Delta}_k = \tilde{m}_k(\sigma)$, and $\tilde{S} = \cup \tilde{\Delta}_k$. The surface $S$ will often be replaced by $\tilde{S}$ in our calculations.

The above discussion has been only for the case of piecewise quadratic interpolation. Polynomial interpolation of other degrees can be used; for example, piecewise linear interpolation of the surface $S$ is very common, along with piecewise constant interpolation for the definition of the collocation method. We have used quadratic interpolation since it illustrates a higher order case than is ordinarily used in practice and has convergence results that are of a higher order than would ordinarily be expected. The results in [14] and [15] can be generalized to handle other degrees of interpolation.

**Numerical integration and the Nyström method.** To evaluate the surface integrals over $S$, we use formulas for numerical integration over $\sigma$:

\[\int_{\sigma} g(s, t) ds \, dt \approx \sum_{j=1}^{r} \omega_j g(s_j, t_j).\]
As an important illustrative case, consider the formula based on integrating the quadratic polynomial that interpolates \( g \) at \( \{q_1, \ldots, q_6\} \):

\[
(2.12) \quad \int_{\sigma} g(s, t) ds \ dt \approx \frac{1}{6} \sum_{j=4}^{6} g(q_j).
\]

This formula has degree of precision two.

When (2.12) is applied to a general triangulation of \( S \), we obtain a composite formula

\[
(2.13) \quad I(G) \equiv \int_{S} G(Q) \ dS_{Q} \approx \sum_{i=1}^{N} w_{i} G(v_{i}).
\]

In this case, we have applied (2.12) with \( g(s, t) = G(m_{k}(s, t))|D_{S}m_{k} \times D_{m_{k}}| \) for each \( \Delta_{k} \in T_{N} \). Actually, this formula involves only the midpoints of the sides of the triangulation, and this means that only \( \frac{3}{2} N \) of the weights in (2.13) are nonzero, a savings of about 25\% in function evaluations. For the error, we can show that

\[
(2.14) \quad I(G) - I_{N}(G) = O(h^{3}).
\]

This result can be improved when the refinement process for the triangulation is defined in a special way. For a general triangle \( \tilde{\Delta} = \tilde{\Delta}_{i,k} \subset R_{i} \), we refine it by connecting the midpoints of its sides. Doing this with all elements of the triangulation means that \( N \) increases by a factor of four with each refinement of the triangulation. Also, we consider replacing \( S \) by \( \tilde{S} \) to obtain the integration formula

\[
(2.15) \quad \int_{S} G(Q) \ dS_{Q} \approx \sum_{k=1}^{N} \sum_{j=4}^{6} G(v_{k,j})|(D_{r}m_{k} \times D_{t}m_{k})(q_{j})|
\]

\[
= \tilde{I}_{N}(G) \equiv \sum_{i=1}^{N} \tilde{w}_{i} G(v_{i})
\]

with \( \tilde{w}_{i} \equiv \tilde{w}_{i}^{(N)} \). With the refinement process we have described, it has been shown by Chien [14] that for \( G \) sufficiently differentiable on each section \( S_{j} \) of \( S \), the quadrature error satisfies

\[
(2.16) \quad I(G) - \tilde{I}_{N}(G) = O(h^{4}).
\]

We apply (2.15) to the solution of the integral equation (2.1) in the case that the kernel function \( K(P, Q) \) is sufficiently differentiable. Nyström’s method is used with the numerical integration operator

\[
(2.17) \quad \mathcal{K}_{N} u(P) = \sum_{i=1}^{N} \tilde{w}_{i} K(P, v_{i}) u(v_{i}), \quad P \in S, \quad u \in C(S).
\]

In (2.1), \( \mathcal{K} \) is replaced by \( \mathcal{K}_{N} \), and we solve the approximating equation \( (\lambda - \mathcal{K}_{N}) u_{N} = g \). This amounts to solving the linear system

\[
(2.18) \quad \lambda u_{N}(v_{i}) - \sum_{j=1}^{N} \tilde{w}_{j} K(v_{i}, v_{j}) u_{N}(v_{j}) = g(v_{i}), \quad i = 1, \ldots, N.
\]
The solution is extended to other points by using the Nyström interpolation formula

\[
(2.19) \quad u_N(P) = \frac{1}{\lambda} \left[ g(P) + \sum_{j=1}^{N_u} \bar{w}_j K(P, v_j) u_N(v_j) \right].
\]

A complete theory of Nyström’s method is given in [3, Part II, Chap. 3] and we omit it here. Using this theory, and using the results of Chien [14], we obtain

\[
(2.20) \quad \|u - u_N\|_\infty = O(h^4).
\]

For a further development and numerical illustrations, see [14] and [15].

**Collocation methods.** The collocation method for solving (2.1) is given by

\[
(2.21) \quad (\lambda - \mathcal{P}_N K) u_N = \mathcal{P}_N g.
\]

We write

\[
(2.22) \quad u_N(P) = \sum_{j=1}^{6} u_N(v_{k,j}) \ell_j(s, t), \quad P = m_k(s, t) \in \Delta_k
\]

for \( 1 \leq k \leq N \). The values \( u_N(v_{k,j}) \) are obtained by solving the system

\[
(2.23) \quad \lambda u_N(v_i) - \sum_{k=1}^{N} \sum_{j=1}^{6} u_N(v_{k,j}) \int_{\sigma} K(v_i, \bar{m}_k(s, t)) \ell_j(s, t) |D_n \bar{m}_k \times D_t \bar{m}_k|(s, t) d\sigma = g(v_i),
\]

\[ i = 1, \ldots, N_v. \]

Note that we are using the approximate surface \( \tilde{S} \), so that we should replace \( K \) in (2.21) with an integral operator \( \tilde{K}_N \) based on integration over \( \tilde{S} \). Thus we would write

\[
(2.24) \quad (\lambda - \mathcal{P}_N \tilde{K}_N) u_N = \mathcal{P}_N g.
\]

For \( S \) a smooth surface, we assume that the kernel function \( K(P, Q) \) has a smooth extension to points \( Q \) in a nearby neighborhood of \( S \); and for \( S \) piecewise smooth, we assume that \( K(P, Q) \) has a smooth extension to points \( Q \) in a neighborhood of each section \( S_i \) of the original surface. This assumption can be justified rigorously, but we omit it here. For larger values of \( N \), the calculation of the coefficients in (2.23) is often the most time-consuming part of the collocation method.

For a solvability theory for collocation methods, see [3], and for the method at hand, see [6] and [7]. In [14], it is shown that

\[
(2.25) \quad \max_{v \in \mathcal{V}_N} |u(v) - u_N(v)| = O(h^4)
\]

provided that \( u \) is sufficiently smooth. In contrast, the standard theory for collocation methods states that

\[
(2.26) \quad \|u - u_N\|_\infty = O(\|u - \mathcal{P}_N u\|_\infty),
\]

which from (2.8) is only \( O(h^3) \). Thus the node points \( v \in \mathcal{V}_N \) are points of superconvergence of the solution \( u_N \).

The integrals in (2.23) are often quite time consuming to evaluate numerically, and hence the Nyström method of (2.18)-(2.19) is often preferable if the kernel function \( K(P, Q) \) is
sufficiently smooth. For cases in which $K(P, Q)$ is singular, as with our intended applications to boundary integral equations, the numerical evaluation of the integrals in (2.23) is discussed at length in [9] and [11], and empirical investigations of orders of convergence are given there. Those numerical methods will be used in the applications of §4 and §5.

**Numerical example.** Consider solving (1.1) with the true solution being

\[ u(x, y, z) = \frac{1}{r} e^{x/y^2} \cos \left( \frac{z}{r^2} \right), \quad r = \sqrt{x^2 + y^2 + z^2}. \]

Let $S$ be the ellipsoidal surface

\[ \left( \frac{x}{a} \right)^2 + \left( \frac{y}{b} \right)^2 + \left( \frac{z}{c} \right)^2 = 1 \]

with $(a, b, c) = (2, 2.5, 3)$. We use the above collocation method (2.24) to solve the boundary integral equation (1.3). For the normal derivative in (1.3), the approximate surface $\tilde{S}$ is used to calculate an approximation to the unit normal

\[ \tilde{v}_Q \approx \tilde{v}_Q \equiv \frac{(D_x \tilde{m}_k \times D_t \tilde{m}_k)(s, t)}{|(D_x \tilde{m}_k \times D_t \tilde{m}_k)(s, t)|}, \quad Q = m_k(s, t), \quad Q \in \Delta_k. \]

Tables 1 and 2 contain numerical results on the error and its empirical rate of convergence to zero. For the column headings, let

\[
\text{Error} = E_N \equiv \max_{v \in V_N} |u(v) - u_N(v)|, \\
\text{Order} = p \equiv \log \left[ \frac{E_{N/4}}{E_N} \right].
\]

At most node points, the order of convergence appears more clearly to be $O(h^4)$; thus it is likely that the values of Order in the tables are approaching four. The theory for this is not yet complete; so far, we have been able to predict an order of only two. (When the true surface $S$ is used, rather than the approximate surface $\tilde{S}$, it is shown in [11] that the maximum error at the node points is $O(h^4 \log(h))$.)

**Table 1.**

<table>
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<th>$N$</th>
<th>$N_v$</th>
<th>Error</th>
<th>Order</th>
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</thead>
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<tr>
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<td>18</td>
<td>1.740E-2</td>
<td></td>
</tr>
<tr>
<td>32</td>
<td>66</td>
<td>2.849E-3</td>
<td>2.61</td>
</tr>
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<td>128</td>
<td>258</td>
<td>2.524E-4</td>
<td>3.50</td>
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<td>512</td>
<td>1026</td>
<td>2.004E-5</td>
<td>3.65</td>
</tr>
</tbody>
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**Table 2.**

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<th>$N_v$</th>
<th>Error</th>
<th>Order</th>
</tr>
</thead>
<tbody>
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<td></td>
</tr>
<tr>
<td>80</td>
<td>120</td>
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<tr>
<td>320</td>
<td>480</td>
<td>6.757E-5</td>
<td>3.37</td>
</tr>
</tbody>
</table>

**3. The two-grid method for Nyström’s method.** We consider the iterative solution of the linear system associated with $(\lambda - K_M)u_M = g$, with $K_M$ defined as in (2.17). The linear system to be solved is
\[ (3.1) \]
\[ \lambda u_M (v_i^{(M)}) - \sum_{j=1}^{M} w_j^{(M)} K (v_i^{(M)}, v_j^{(M)}) u_M (v_j^{(M)}) = g (v_i^{(M)}), \quad i = 1, \ldots, M, \]

where the superscript is included to show explicitly the dependence of the weights and nodes on \( M \). If the order \( M_v \) is relatively large, say \( M_v \geq 1000 \), then on most computers this system must be solved iteratively. Let \( N \) be less that \( M \), generally much less, and assume that the linear system associated with \((\lambda - K_N)u_N = g\) can be solved directly. We use the explicit knowledge of \((\lambda - K_N)^{-1}\) to iteratively solve (3.1).

The following two-grid iteration is discussed in detail in [3, Part 2, Chap. 4], and we merely summarize some of those results for the present case. For the operator equation, the iteration is as follows. We assume that an initial guess \( u_M^{(0)} \) is given.

1. \[ r^{(l)} = g - (\lambda - K_M) u_M^{(l)}, \]
2. \[ p^{(l)} = K_M r^{(l)}, \]
3. \[ d^{(l)} = (\lambda - K_N)^{-1} p^{(l)}, \]
4. \[ u_M^{(l+1)} = u_M^{(l)} + \frac{1}{\lambda} [r^{(l)} + d^{(l)}]. \]

These relations can be specialized to the linear system (3.1) and the analogous linear system for parameter \( N \); the Nyström interpolation formula (2.19) furnishes the means of connecting the systems of order \( M_v \) and \( N_v \).

It can be shown that the iterates satisfy the error equation

\[ (3.2) \quad u_M - u_{M}^{(l+1)} = \frac{1}{\lambda} (\lambda - K_N)^{-1} (K_M - K_N) K_M (u_M - u_M^{(l)}). \]

It can be shown that \( \{K_M | M \geq 1\} \) is a collectively compact and pointwise convergent family of operators on \( C(S) \); consequently,

\[ (3.3) \quad \limsup_{N \to \infty} \sup_{M \geq N} \| (K_m - K_N) K_M \| = 0. \]

This shows that

\[ (3.4) \quad \frac{1}{\lambda} (\lambda - K_N)^{-1} (K_M - K_N) K_M < 1 \]

for all sufficiently large \( N \), uniformly for \( M > N \). More can be said about this two-grid method, but we refer the reader to [2] or [3].

**Numerical example.** Consider the integral equation

\[ (3.5) \quad \lambda u(P) - \int_S u(Q) \frac{\partial}{\partial Q} (|P - Q|^2) dS_Q = g(P), \quad P \in S \]

with \( S \) the ellipsoidal surface (2.34) and \((a, b, c) = (1.0, .75, .5)\). The right-hand function \( g(P) \) is chosen so that the true solution is \( u(x, y, z) = e^z \). In Table 3, we give iteration convergence rates for various values of \( N, M, \) and \( \lambda \). This rate is defined by

\[ (3.6) \quad \text{Ratio} = \lim_{l \to \infty} \frac{\|u_M^{(l)} - u_M^{(l-1)}\|_\infty}{\|u_M^{(l-1)} - u_M^{(l-2)}\|_\infty}, \]

which we compute empirically. In all cases, such a limit existed computationally. The numerical results illustrate that the value of Ratio should approach a constant as \( M \to \infty, \)
which can be proven from (3.2). Also note that the values of Ratio decrease substantially when $N$ is increased.

The norm on $C(S)$ of the integral operator $\mathcal{K}$ of (3.5) used in this example is $\|\mathcal{K}\| = 3\pi$. The variation in convergence rates between $\lambda = 30$ and $\lambda = 10$ shows the effect of choosing $\lambda$ closer to the spectrum of $\mathcal{K}$. As $\lambda$ approaches the spectrum of $\mathcal{K}$, the size of $\| (\lambda - \mathcal{K})^{-1} \|$ increases in size and thus the linear multiplying factor in the iteration error equation (3.2) also increases in size.

4. Two-grid iteration for the collocation method. We modify the two-grid iteration of §3, in line with ideas in [19] and [20]. In the two-grid iteration I1–I4 of §3, we moved between coarse grid functions and fine grid functions by means of a Nyström interpolation, as in (2.19). Here we use a scheme of prolongation and restriction operators.

The linear systems associated with

\begin{equation}
(\lambda - P_M \tilde{K}_M)u_M = P_M g
\end{equation}

and

\begin{equation}
(\lambda - P_N \tilde{K}_N)u_N = P_N g
\end{equation}

have orders $M_0$ and $N_0$, respectively, and are associated with triangulations $T_M = \{ \Delta_k^{(M)} \}$ and $T_N = \{ \Delta_k^{(N)} \}$, respectively. We need a way to move between functions defined on the grids $\mathcal{V}_M = \{ v_k^{(M)} \}$ and $\mathcal{V}_N = \{ v_k^{(N)} \} \subset \mathcal{V}_M$.

Define a restriction operator

$$
\mathcal{R}_{MN} : \mathbb{R}^{M_0} \to \mathbb{R}^{N_0}
$$

by

\begin{equation}
(\mathcal{R}_{MN} \rho_M) \left( v_k^{(N)} \right) = \rho_M \left( v_k^{(N)} \right), \quad 1 \leq k \leq N_0, \quad \rho_M \in \mathbb{R}^{M_0}.
\end{equation}

In this, $\rho_M$ is regarded as a function on the grid $\mathcal{V}_M$, and $\mathcal{R}_{MN} \rho_M$ is regarded as a function on the grid $\mathcal{V}_N$. 

\begin{table}
\centering
\caption{Convergence rates for two-grid iteration with system (3.1) for (3.5).}
\begin{tabular}{cccc}
\hline
$\lambda$ & $N$ & $M$ & Ratio \\
\hline
30 & 8 & 32 & .0119 \\
 & 8 & 128 & .0138 \\
 & 8 & 512 & .0140 \\
 & 32 & 128 & .0018 \\
 & 32 & 512 & .0020 \\
 & 20 & 80 & .0039 \\
 & 20 & 320 & .0043 \\
 & 80 & 320 & .00037 \\
\hline
10 & 8 & 32 & .5006 \\
 & 8 & 128 & .5817 \\
 & 8 & 512 & .5889 \\
 & 32 & 128 & .1616 \\
 & 32 & 512 & .1760 \\
\hline
\end{tabular}
\end{table}
Define a prolongation operator
\[ P_{NM} : \mathbb{R}^{N_0} \rightarrow \mathbb{R}^{M_0} \]
as follows. Let \( \rho_N \in \mathbb{R}^{N_0} \) be a function on the coarse grid \( \mathcal{V}_N \), which we wish to extend to a function on \( \mathcal{V}_M \). Let \( v_{k,j}^{(M)} \in \mathcal{V}_M \). Then \( v_{k,j}^{(M)} \in \Delta^k_i^{(M)} \), and this triangular element is contained in a unique element \( \Delta^l_i^{(N)} \) for some \( 1 \leq l \leq N \). There is
\[ m^{(N)}_i : \sigma_{1-1}^{(N)} \]
Because \( \Delta^k_i \subset \Delta^l_i \), the point \( v_{k,j}^{(M)} = m^{(N)}_i (\hat{s}, \hat{t}) \) for some \( (\hat{s}, \hat{t}) \in \sigma \). Define
\[ (P_{NM} \rho_N) \left( v_{k,j}^{(M)} \right) = \sum_{i=1}^6 \rho_N \left( v_{i,j}^{(N)} \right) \ell_i(\hat{s}, \hat{t}). \]
When refining a triangulation, it is best to calculate and save the needed values of \( (\hat{s}, \hat{t}) \) while doing the refinement.

Write the coarse grid linear system (2.23), associated with (4.2), as
\[ (\lambda I - K_N)u_N = g_N, \quad u_N \in \mathbb{R}^{N_0}, \]
and write the corresponding fine grid linear system associated with (4.1) as
\[ (\lambda I - K_M)\mu_M = g_M, \quad \mu_M \in \mathbb{R}^{M_0}. \]
The two-grid iteration for solving the fine grid system is as follows.

1. \( r^{(0)} = g_M - (\lambda I - K_M)u_M^{(0)} \),
2. \( p^{(0)} = R_{MN}K_Mr^{(0)} \),
3. \( d^{(0)} = P_{NM}(\lambda I - K_N)^{-1}p^{(0)} \),
4. \( \mu^{(0)} = \mu_M^{(0)} + \frac{1}{\lambda} \left[ r^{(0)} + d^{(0)} \right] \).

An equivalent form can be given that reminds one of multigrid iteration.

1. \( \hat{\mu}_M^{(0)} = \frac{1}{\lambda} [g_M + K_M\mu_M^{(0)}] \),
2. \( \hat{r}^{(0)} = g_M - (\lambda I - K_M)\hat{\mu}_M^{(0)} \),
3. \( \mu^{(0)} = \hat{\mu}_M^{(0)} + P_{NM}(\lambda I - K_N)^{-1}R_{MN}r^{(0)} \).

Step 1 is called the “smoothing step” and step 3 is called the “correction step.” The replacement of Nyström interpolation, as in (2.19), with the quadratic interpolation of (4.4) yields a less expensive way to move from a coarse grid function to a fine grid function following ideas in [19].

For the iteration error, it is straightforward to derive
\[ \mu_M - \mu^{(0)} = C_{N,M} \left[ \mu_M^{(0)} - \mu_M^{(0)} \right], \]
and
\[ C_{N,M} = \frac{1}{\lambda} \left[ I - P_{NM}(\lambda I - K_N)^{-1}R_{MN}(\lambda I - K_M) \right] K_M. \]
To have convergence of the iteration, one must show that
\[ \| C_{N,M} \| < 1. \]
We consider here only the type of integral operators arising in boundary integral equations as in (1.3).

If \( S \) is a smooth surface, then the integral operator \( \mathcal{K} \) of (1.3) is compact on \( C(S) \) into \( C(S) \); the same is true of other integral operators of potential theory with a weakly singular kernel. From this, we can use the tools associated with the theory of collectively compact operators to show that

\[
\text{Limit Sup } \left\| C_{N,M} \right\| = 0.
\]

The proof is given in the Appendix of this paper.

**Numerical example.** Consider the BIE (1.3) with the known true solution (2.27), and use the same ellipsoidal surface \( S \) of (2.28). We consider the iterative solution of the linear systems for this earlier example. To save time in carrying out the study of the iteration method, we did not evaluate the collocation integrals

\[
\int_{\sigma} K(v_i, \tilde{m}_k(s, t)) \ell_j(s, t)(D_i \tilde{m}_k \times D_j \tilde{m}_k)(s, t) d\sigma
\]

of (2.23) as accurately as had been done in obtaining the values in Tables 1 and 2. For this section, the above integrals were divided into two classes. For \( v_i \in \Delta_k \), a change of variables was performed to eliminate the singularity and then a low-order product Gaussian quadrature formula was used. For \( v_i \in \Delta_k \), the integration was performed using formula T2:5–1 of Stroud [35, p. 314]. This is a seven-point formula with degree of precision five. Empirically, we have found that there is very little difference in the performance of the iteration methods for the system obtained with the quadrature just described and with the linear system (2.23) using very accurate evaluation of all collocation integrals. For a further discussion of the evaluation of the collocation integrals, see [7], [9], [11], [34], and [17].

Table 4 contains the empirical iteration rates of (3.6). Note again that the rate of convergence improves as the coarse mesh parameter \( N \) is increased. This agrees with (4.10). For one case (indicated by †), the empirical limit in (2.41) did not occur, and we used a geometric average of the final few ratios to obtain the value given in the table. We also see in the table that for fixed \( N \), the convergence ratios appear to be approximately constant as \( M \) increases.

**Further discussion.** In [21] and [31] methods are given for the rapid (approximate) evaluation of the matrix-vector product \( K_N \rho \), \( \rho \in \mathbb{R}^N \). The cost of straightforward evaluation
is approximately $2N^2$ arithmetic operations. The cost in [21] is $O(N_v \log^d N_v)$ arithmetic operations; the method in [31] is similar in cost. Since the evaluation of $K_N \rho$ is a major part of the cost, and since the preceding authors also give a less expensive way to evaluate the matrix $K_N$, their methods can also be used with the present two-grid method to further reduce its cost. Alternatively, we can consider the use of vector/parallel processors in evaluating $K_N \rho$, which will also reduce the cost of the iteration. Regardless of how $K_N \rho$ is evaluated, the discussion of the two-grid iteration presented here remains valid as it is independent of the method of evaluating $K_N \rho$. For the case that $S$ is a smooth surface, the two-grid iteration of this section seems to be an easy-to-use method with good convergence.

5. Iteration for BIEs on piecewise smooth surfaces. When the surface $S$ is only piecewise smooth, the double-layer integral operator $K$ of (1.3) is not compact on $C(S)$, although it is still bounded. Consequently, the standard theory of collectively compact operator approximations ([1]) does not apply in this case, and a convergence theory must be developed in some other way. For the limited convergence theory of collocation methods that is available for (1.3) when $S$ is only piecewise smooth, see [36], [37], [23], [6], [7], [9], and [11].

The given theories depend on dividing the surface into two subsets, one subset of points “close” to edges and corners of $S$ and the other subset the remaining part of $S$. The integral equation (1.4) is written as

\begin{equation}
2\pi \left[ \begin{array}{c}
u_1 \\ u_2 \end{array} \right] + \left[ \begin{array}{cc}
K_{11} & K_{12} \\
K_{21} & K_{22} \end{array} \right] \left[ \begin{array}{c}
u_1 \\ u_2 \end{array} \right] = \left[ \begin{array}{c}g_1 \\ g_2 \end{array} \right].
\end{equation}

In this, all functions $u \in C(S)$ are decomposed as

\begin{equation}
u_1 = u|_{S_1}, \quad u_2 = u|_{S_2}
\end{equation}

and for the function space, we use $X = C(S_1) \oplus C(S_2)$. Following this decomposition, the integral operators $K_{ij}$ are defined in a natural way from $C$, using the decomposition of $S$, and

$K_{ij} : C(S_j) \to C(S_i)$.

For the solvability theory for (1.3) and the convergence theory for numerical methods for solving (1.3), the invertibility of $2\pi + K_{11}$ and its numerical approximants is established first. After doing this, the remaining part of the problem can be attacked using standard tools based on compactness, since $K_{ij}$ is compact for $(i, j) \neq (1, 1)$. This approach has been used for planar BIEs on regions with a piecewise smooth boundary curve and is still the main way of approaching analogous equations in three dimensions. This approach to the problem is due originally to Radon [30] for planar problems, and it was extended to three-dimensional BIEs and their numerical solution by Wendland [36].

For the collocation method proposed in (2.24) in \S 2, we assume that the surface $S$ satisfies the same properties as those assumed in [36]. With these properties, we obtain convergence of the numerical method (2.26) if

\begin{equation}
\frac{5}{3} |2\pi - \Omega(P)| < 1
\end{equation}

at all points $P \in S$ [6], [7]. The assumption (5.3) seems to be an artificial one based on the method of analysis being used. Empirically, (5.3) appears to be unnecessary.

In addition to the problems of the invertibility of $2\pi + K$, there is also the problem of dealing with the behaviour of the solution $u(P)$ for points $P$ near to edges and corners of $S$. 
It is known from the planar theory that the function \( u(P) \) usually has algebraic singularities that must be considered in the numerical solution of the equation. For a development of ideas in this direction for (1.3), see [16] and [29].

The two-grid iteration method. Because \( \mathcal{K} \) is not compact, the condition (4.10) is not true and there is no reason to believe that the convergence requirement (4.9) is true. We have also studied the analogous iteration for planar BIEs in [13], and when the method converged, the rate of convergence did not depend on the size of \( N \). We have investigated empirically the two-grid iteration \( J_1-J_4 \) and we give some of the results of that study here.

Let \( S \) be the elliptical paraboloid

\[
\left( \frac{x}{a} \right)^2 + \left( \frac{y}{b} \right)^2 = z, \quad 0 \leq z \leq c
\]  

(5.4a)

together with the “cap” of points \((x, y, z)\) satisfying

\[
\left( \frac{x}{a} \right)^2 + \left( \frac{y}{b} \right)^2 \leq c, \quad z = c.
\]  

(5.4b)

We again solved (1.3) and the true solution was chosen to be

\[
u(x, y, z) = [(x - \frac{1}{a})^2 + y^2 + z^2]^{-1/2}.
\]

Results for the two-grid iteration method are given in Table 5.

The first observation is that the two-grid iteration still works, provided that \( N \) is sufficiently large. Second, the rate of convergence does not improve as \( N \) increases (provided that \( N \) is large enough to have Ratio \(< 1\)), contrary to the case with a smooth surface.

**Table 5**

<table>
<thead>
<tr>
<th>((a, b, c))</th>
<th>(N)</th>
<th>(M)</th>
<th>Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>(.5, .5, 1)</td>
<td>8</td>
<td>32</td>
<td>.615</td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>128</td>
<td>.592</td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>512</td>
<td>.586</td>
</tr>
<tr>
<td></td>
<td>32</td>
<td>128</td>
<td>.594</td>
</tr>
<tr>
<td></td>
<td>32</td>
<td>512</td>
<td>.586</td>
</tr>
<tr>
<td>(1, 1, 1)</td>
<td>8</td>
<td>32</td>
<td>.668</td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>128</td>
<td>.658</td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>512</td>
<td>.652</td>
</tr>
<tr>
<td>(2, 2, 1)</td>
<td>8</td>
<td>32</td>
<td>.761</td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>128</td>
<td>.758</td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>512</td>
<td>.754</td>
</tr>
<tr>
<td></td>
<td>32</td>
<td>128</td>
<td>.757</td>
</tr>
<tr>
<td></td>
<td>32</td>
<td>512</td>
<td>.754</td>
</tr>
<tr>
<td>(2.5, 2.5, 1)</td>
<td>8</td>
<td>32</td>
<td>89.326</td>
</tr>
<tr>
<td></td>
<td>32</td>
<td>128</td>
<td>.790†</td>
</tr>
<tr>
<td></td>
<td>32</td>
<td>512</td>
<td>.862</td>
</tr>
</tbody>
</table>

For the paraboloid surface of (5.4) with \( a = b \), the interior solid angle on the edge of \( S \) at \( z = c \) is

\[
\Omega_a = 2 \tan^{-1} \left( \frac{2\sqrt{c}}{a} \right).
\]  

Using Table 5 results and this formula, we find that for the convergence ratio \( r_a \equiv \text{Ratio} \), most of the values of \( r_a \) satisfy

\[
r_a \doteq 1 - .157\Omega_a.
\]  

(5.6)
This makes some sense since

\[ 1 - 0.157 \Omega_d \approx 0 \Rightarrow \Omega_d \approx 2\pi, \]

and that would correspond to a smooth surface. With a smooth surface, the earlier result \((4.10)\) implies that the value of \(\text{Ratio} \) converges to zero as \(N \to \infty\).

A modified two-grid iteration. The split \((5.1)\) is required for not only a convergence analysis of the numerical solution of \((2\pi + K)u = g\), but also seems to be necessary when iteratively solving the linear system \((2\pi + K_M)\mu_M = g_M\). This has been noted previously in \([32], [33], [23], \) and \([13]\). We use this split to obtain a modified linear system for which two-grid iteration performs better.

Based on the split \((5.1)\), we make an analogous splitting of the domain of the grid function \(\mu_M\). Then \((2\pi + K_M)\mu_M = g_M\) is rewritten as

\[
(5.7) \quad 2\pi \left[ \begin{array}{c} \mu_{1M}^1 \\ \mu_2^2 \end{array} \right] + \left[ \begin{array}{cc} K_{11}^{11} & K_{12}^{12} \\ K_{21}^{11} & K_{22}^{12} \end{array} \right] \left[ \begin{array}{c} \mu_{1M}^1 \\ \mu_2^2 \end{array} \right] = \left[ \begin{array}{c} g_1^1 \\ g_2^2 \end{array} \right].
\]

We assume that \(K_{11}^{11}\) has been so constructed that \(2\pi + K_{11}^{11}\) is nonsingular. Then \((5.7)\) is rewritten as

\[
(5.8) \quad \left[ \begin{array}{cc} 2\pi & 2\pi [2\pi + K_{11}^{11}]^{-1} K_{12}^{22} \\ K_{21}^{11} & 2\pi + K_{22}^{22} \end{array} \right] \left[ \begin{array}{c} \mu_{1M}^1 \\ \mu_2^2 \end{array} \right] = \left[ \begin{array}{c} 2\pi [2\pi + K_{11}^{11}]^{-1} g_1^1 \\ g_2^2 \end{array} \right].
\]

Denote this linear system by

\[
(5.9) \quad (2\pi + L_M)\mu_M = g_M, \quad M \geq 1.
\]

We solve this by using two-grid iteration, using a lower order system

\[ (2\pi + L_N)\mu_N = g_N \]

as our coarse grid system.

How is the partition of the domain \(\mathcal{V}_M\) of the grid function \(\mu_M\) to be made? The usual definition is to choose a small \(\varepsilon > 0\) and to then define

\[
\mathcal{V}_M^1 = \{ v_i \mid \text{the distance from } v_i \text{ to an edge or vertex of } S \text{ is } \leq \varepsilon \},
\]

\[
\mathcal{V}_M^2 = \mathcal{V}_M \backslash \mathcal{V}_M^1.
\]

Then for \(\mu \in \mathbb{R}^{M^0}\), define

\[
(5.11) \quad \mu^1 = \mu|\mathcal{V}_M^1, \quad \mu^2 = \mu|\mathcal{V}_M^2.
\]

This type of scheme is used in both \([32]\) and \([13]\) for planar BIE problems and leads to efficient iteration methods. The value of \(\varepsilon\) should be small enough to have the order of \(2\pi + K_{11}^{11}\), called \(D_M\), be small enough to directly solve linear systems with \(2\pi + K_{11}^{11}\) as the coefficient matrix. The order \(D_M = O(\varepsilon M)\) and, as \(M\) increases, \(D_M\) increases at the same rate as for the original system. For planar BIE problems, \(D_M\) is generally sufficiently small.

For three-dimensional systems, \(D_M\) increases much more rapidly and the associated linear system can easily become too large to be handled directly. To avoid having a system with a large order, we have experimented using \(\varepsilon = 0\). Thus \(v_i \in \mathcal{V}_M^1\) if and only if \(v_i\) is a vertex of
the original surface $S$ or is on an edge of $S$. This ensures that $D_M = O(\sqrt{M})$. It still grows with $M$, but generally remains manageable. When $\epsilon > 0$, we can prove convergence of the iteration method in some cases, but we have no proof for the case $\epsilon = 0$, although it does give significantly improved results over the two-grid method for the original linear system $(2\pi + K_M)\mu_M = g_M$.

**Example.** We repeat the solution of the linear systems used in obtaining Table 5. The new results are given in Table 6. In every case in the table, the iteration ratios

$$R_l \equiv \frac{\|\mu_M^{(l)} - \mu_M^{(l-1)}\|_\infty}{\|\mu_M^{(l-1)} - \mu_M^{(l-2)}\|_\infty}$$

behaved in a somewhat unusual manner, oscillating in a small interval without converging to a limit. In Table 6, we give the “limiting mean” $\bar{R}$ of those oscillating values. In Fig. 1, we show these iteration ratios $R_l$ for the case

$$(a, b, c) = (2.5, 2.5, 1), \quad N = 8$$

with three values of $M$. The iteration was carried out until the estimated iteration error was less than $10^{-12}$. We also give the number of iterates $NI$ needed to satisfy the test

$$\|\mu_M^{(l)} - \mu_M^{(l-1)}\|_\infty \leq 10^{-8}$$

when beginning with $\mu_M^{(0)} = 0$.

**Example.** Solve problem (1.1) on a solid simplex with vertices

$$(0, 0, 0), (a, 0, 0), (0, b, 0), (0, 0, c).$$

The true solution was chosen to be

$$u(x, y, z) = \frac{1}{|x, y, z| - \frac{1}{4}(a, b, c)}.$$

The values of $\bar{R}$ are given in Table 7 for various values of $(a, b, c), N$, and $M$. The simplexes with $(a, b, c) = (1, 1, 1)$ and $(3, 3, 3)$ gave essentially the same results on the rate of convergence of the iteration method. In contrast with the case with $S$ as a paraboloid, the iteration
ratios $R_l$ often converged to a limiting value. This was especially true when $M/N > 4$, but not always. Figure 2 contains graphs of the logarithms of the iteration corrections $\|\mu^{(l)}_M - \mu^{(l-1)}_M\|_{\infty}$ as $l$ increases. The parameters for $S$ in this figure are

$$(a, b, c) = (1, 1, 3), \quad N = 4, \quad M = 16, 64, 256.$$  

From Fig. 2 and analogous figures for other values of the problem parameters, it is clear that the iteration is converging linearly, with the rate becoming worse as $M$ increases.

### Table 7

*Modified two-grid iteration for (1.3) on a simplex.*

<table>
<thead>
<tr>
<th>$(a, b, c)$</th>
<th>$N$</th>
<th>$N_v$</th>
<th>$M$</th>
<th>$M_v$</th>
<th>$D_M$</th>
<th>$\bar{R}$</th>
<th>$NI$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(3, 3, 3)$</td>
<td>4</td>
<td>10</td>
<td>16</td>
<td>34</td>
<td>22</td>
<td>.29</td>
<td>18</td>
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<tr>
<td></td>
<td>4</td>
<td>10</td>
<td>64</td>
<td>130</td>
<td>46</td>
<td>.34</td>
<td>20</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>10</td>
<td>256</td>
<td>514</td>
<td>94</td>
<td>.55</td>
<td>31</td>
</tr>
<tr>
<td>$(1, 1, 1)$</td>
<td>4</td>
<td>10</td>
<td>16</td>
<td>34</td>
<td>22</td>
<td>.29</td>
<td>19</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>10</td>
<td>64</td>
<td>130</td>
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<td>.35</td>
<td>21</td>
</tr>
<tr>
<td></td>
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<td>10</td>
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<td>514</td>
<td>94</td>
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<td>33</td>
</tr>
<tr>
<td></td>
<td>16</td>
<td>34</td>
<td>64</td>
<td>130</td>
<td>46</td>
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<td>17</td>
</tr>
<tr>
<td></td>
<td>16</td>
<td>34</td>
<td>256</td>
<td>514</td>
<td>94</td>
<td>.43</td>
<td>25</td>
</tr>
<tr>
<td>$(1, 1, 2)$</td>
<td>4</td>
<td>10</td>
<td>16</td>
<td>34</td>
<td>22</td>
<td>.34</td>
<td>20</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>10</td>
<td>64</td>
<td>130</td>
<td>46</td>
<td>.55</td>
<td>33</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>10</td>
<td>256</td>
<td>514</td>
<td>94</td>
<td>.69</td>
<td>51</td>
</tr>
<tr>
<td></td>
<td>16</td>
<td>34</td>
<td>64</td>
<td>130</td>
<td>46</td>
<td>.35</td>
<td>20</td>
</tr>
<tr>
<td></td>
<td>16</td>
<td>34</td>
<td>256</td>
<td>514</td>
<td>94</td>
<td>.54</td>
<td>31</td>
</tr>
<tr>
<td>$(1, 1, 3)$</td>
<td>4</td>
<td>10</td>
<td>16</td>
<td>34</td>
<td>22</td>
<td>.42</td>
<td>24</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>10</td>
<td>64</td>
<td>130</td>
<td>46</td>
<td>.62</td>
<td>42</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>10</td>
<td>256</td>
<td>514</td>
<td>94</td>
<td>.74</td>
<td>65</td>
</tr>
<tr>
<td></td>
<td>16</td>
<td>34</td>
<td>64</td>
<td>130</td>
<td>46</td>
<td>.42</td>
<td>24</td>
</tr>
<tr>
<td></td>
<td>16</td>
<td>34</td>
<td>256</td>
<td>514</td>
<td>94</td>
<td>.62</td>
<td>35</td>
</tr>
</tbody>
</table>

6. **Conclusions.** Computations were carried out for a number of piecewise smooth surfaces with various choices of boundary data. From this, it is clear that the modified two-grid method based on (5.9) converges linearly. As $N$ increases, the rate of linear convergence...
as measured by $\tilde{R}$ decreases, thus giving an improved rate of convergence for the iteration method. But as $M$ increases for fixed $N$, the ratios $\tilde{R}$ increase in size and the convergence is worse. The iteration based on (5.9) is clearly superior to the use of the unmodified two-grid iteration, as illustrated by comparing Tables 5 and 6.

It appears that $R$ is a function of $M/N$, rather than a function of $N$ and $M$ separately. For constant $M/N$, the empirical rate of linear convergence $\tilde{R}$ is also approximately constant. Thus we need to make $N$ larger to maintain a good rate of convergence for larger values of $M$; of course, this means there is an additional cost in solving the coarse mesh equation. Since we also are solving systems of size $D_M$, it will not mean much of an additional cost to have $N$ be approximately the same size as $D_M$. By doing so, the rate of convergence will not degrade as rapidly when $M$ increases.

If we had used $\epsilon > 0$ in the definition (5.10), then we could have proved that the ratios would behave in the same manner as for the two-grid method on a smooth surface. But in that case, the size of $D_M$ increases much more rapidly, as it is proportional to $M$ rather than to $\sqrt{M}$. Whether it is better to use $\epsilon = 0$ or $\epsilon > 0$ depends on the particular problem being solved and the computer on which it is being solved. Generally it seems better to let $\epsilon = 0$ as long as the iteration is converging, and to turn to $\epsilon > 0$ only if convergence is not being attained.

![Fig. 2. $l$ vs. $-\log_{10} \| \mu_{M}^{(0)} - \mu_{M}^{(0-1)} \|_{\infty}$. Iteration corrections for simplex.](image)

Appendix. We give a proof of the result (4.10):

\begin{equation}
\text{Limit Sup}_{N \to \infty, M > N} \| C_{N,M} \| = 0.
\end{equation}

The proof is divided into two cases: (1) the exact surface $S$ is used in all integrations and (2) the approximate surface $\tilde{S}$ is used. We begin by introducing some additional notation.

Define $\mathcal{R}_{\infty} : C(S) \to \mathbb{R}^{M}$ by

\begin{equation}
(R_{\infty} g)(v^{(M)}_{i}) = g(v^{(M)}_{i}), \quad i = 1, \ldots, M, \quad g \in C(S),
\end{equation}

where elements of $\mathbb{R}^{M}$ are regarded as functions on the grid $\mathcal{V}_{M}$. Define $\mathcal{P}_{M} : \mathbb{R}^{M} \to C(S)$ by

\begin{equation}
(P_{M} \rho)(m_{k}(s, t)) = \sum_{i=1}^{6} \rho(v^{(M)}_{k,i}) \ell_{i}(s, t), \quad (s, t) \in \sigma, \quad k = 1, \ldots, M, \quad \rho \in \mathbb{R}^{M}.
\end{equation}
With these, we have the following identities:

(A.4) \[ \mathcal{R}_{\infty M} \mathcal{P}_{\infty} = I, \]
(A.5) \[ \mathcal{P}_{\infty} \mathcal{R}_{\infty M} = \mathcal{P}_M, \]
(A.6) \[ \mathcal{K}_M = \mathcal{R}_{\infty M} \mathcal{K} \mathcal{P}_{\infty}. \]

We also need the following relations between the coarse and fine grid operators:

(A.7) \[ \mathcal{R}_{MN} \mathcal{R}_{\infty M} = \mathcal{R}_{\infty N}, \]
(A.8) \[ \mathcal{P}_{\infty} \mathcal{P}_{NM} = \mathcal{P}_{N\infty}. \]

The first of these relations is obvious; and the second depends on the uniqueness of quadratic interpolation and the fact that \( T_M \) is a refinement of \( T_N \).

To analyze the iteration \( J_1 - J_4 \) of \( J \), we transform it into an equivalent problem on the space \( C(S) \), thereby eliminating the use of the different spaces \( \mathbb{R}^M \) and \( \mathbb{R}^N \). Introduce

\[ v^{(l)} := \mathcal{P}_{\infty} \mu^{(l)}_M. \]

Recall the collocation equation

(A.9) \[ (I - \mathcal{P}_M \mathcal{K}) u_M = \mathcal{P}_M g \]
and its matrix equivalent

(A.10) \[ (I - \mathcal{K}_M) \mu_M = g_M. \]

The solutions of the two preceding equations satisfy

(A.11) \[ \mu_M = \mathcal{R}_{\infty M} u_M. \]

Using the above in the definitions of \( J_1 - J_4 \) of \( J \), we obtain

\[ \mathcal{P}_{\infty} v^{(l)} = \mathcal{P}_{\infty} g_M - \mathcal{P}_{\infty} (\lambda - \mathcal{R}_{\infty M} \mathcal{K} \mathcal{P}_{\infty}) \mu^{(l)}_M = \mathcal{P}_M g - \lambda v^{(l)} + \mathcal{P}_M K_v^{(l)} = (\lambda - \mathcal{P}_M \mathcal{K})(u_M - v^{(l)}), \]

\[ \mathcal{K}_M v^{(l)} = \mathcal{R}_{\infty M} \mathcal{K}(\lambda - \mathcal{P}_M \mathcal{K}) (u_M - v^{(l)}), \]

\[ d^{(l)} = \mathcal{P}_{NM}(\lambda - K_N)^{-1} \mathcal{R}_{MN} \mathcal{R}_{\infty M} \mathcal{K}(\lambda - \mathcal{P}_M \mathcal{K})(u_M - v^{(l)}) = \mathcal{P}_{NM}(\lambda - K_N)^{-1} \mathcal{R}_{\infty N} \mathcal{K}(\lambda - \mathcal{P}_M \mathcal{K})(u_M - v^{(l)}), \]

\[ v^{(l+1)} = v(l) + \frac{1}{\lambda} \mathcal{P}_{\infty} v^{(l)} + \frac{1}{\lambda} \mathcal{P}_{\infty} d^{(l)} = v^{(l)} + \frac{1}{\lambda} (\lambda - \mathcal{P}_M \mathcal{K})(u_M - v^{(l)}) + \frac{1}{\lambda} \mathcal{P}_{\infty} \mathcal{P}_{NM}(\lambda - K_N)^{-1} \mathcal{R}_{\infty N} \mathcal{K}(\lambda - \mathcal{P}_M \mathcal{K})(u_M - v^{(l)}). \]

Using (A.8), it holds that

(A.12) \[ u_M - v^{(l+1)} = \left\{ I - \frac{1}{\lambda} (\lambda - \mathcal{P}_M \mathcal{K}) - \frac{1}{\lambda} \mathcal{P}_{\infty} (\lambda - K_N)^{-1} \mathcal{R}_{\infty N} \mathcal{K}(\lambda - \mathcal{P}_M \mathcal{K}) \right\} (u_M - v^{(l)}). \]

The following is a well-known identity. If \( (I - BA)^{-1} \) exists, then \( (I - AB)^{-1} \) exists and

\[ (I - AB)^{-1} = I + A(I - BA)^{-1} B. \]
Apply this to convert \((\lambda - K_N)^{-1}\) to an equivalent form. With some straightforward manipulation and use of the earlier identities, we obtain

\[
\mathcal{P}_N \infty (\lambda - K_N)^{-1} R_{\infty N} = \mathcal{P}_N \infty (\lambda - \mathcal{R}_{\infty N} \mathcal{K} \mathcal{P}_{N \infty})^{-1} R_{\infty N} = \frac{1}{\lambda} \mathcal{P}_N (\lambda - \mathcal{K} \mathcal{P}_N)^{-1} = \frac{1}{\lambda} (\lambda - \mathcal{P}_N \mathcal{K})^{-1} \mathcal{P}_N.
\]

Using this result in (A.12) and some straightforward manipulation,

\[
u_M - \nu^{(l+1)} = \mathcal{C}_{N,M} (u_M - \nu^{(l)}),
\]

\[
\mathcal{C}_{N,M} = (\lambda - \mathcal{P}_N \mathcal{K})^{-1} (\mathcal{P}_M - \mathcal{P}_N) \mathcal{K}.
\]

Since \(\mathcal{K}\) is compact on \(C(S)\), it is well known that

\[
\lim_{N \to \infty} \|\mathcal{K} - \mathcal{P}_N \mathcal{K}\| = 0.
\]

Thus \((\lambda - \mathcal{P}_N \mathcal{K})^{-1}\) is uniformly bounded in \(N\) and

\[
\lim_{N \to \infty} \sup_{M > N} \|\mathcal{P}_M \mathcal{K} - \mathcal{P}_N \mathcal{K}\| = 0.
\]

This proves that

\[
\lim_{N \to \infty} \sup_{M > N} \|\mathcal{C}_{N,M}\| = 0,
\]

which shows that the iterates \(\nu^{(l)}\) converge to \(u_M\) as \(l \to \infty\), provided that \(N\) is chosen sufficiently large. As \(M \to \infty\), \(\mathcal{C}_{N,M}\) approaches

\[
(\lambda - \mathcal{P}_N \mathcal{K})^{-1} (I - \mathcal{P}_N) \mathcal{K}.
\]

This, too, is less than one for \(N\) chosen sufficiently large; and thus the convergence in (A.14) will exhibit a “mesh-independence principle.”

To complete the result, these results must be related to the matrix-vector equation (A.1). Write the error equation (A.14) as

\[
e^{(l+1)} = \mathcal{C}_{N,M} e^{(l)}
\]

and the error equation (4.7) as

\[
e^{(l+1)} = \mathcal{C}_{N,M} e^{(l)}.
\]

The errors are connected by the relation

\[
\mathcal{R}_{\infty M} e^{(l)} = e^{(l)}, \quad l \geq 0.
\]

Since \(u_M \in \text{Range}(\mathcal{P}_M)\), we also have

\[
\mathcal{P}_{\infty M} e^{(l)} = \mathcal{P}_{\infty M} \mathcal{R}_{\infty M} e^{(l)} = \mathcal{P}_{M} e^{(l)} = e^{(l)}, \quad l \geq 0.
\]

Using these results with (A.16)

\[
e^{(l+1)} = \mathcal{R}_{\infty M} e^{(l+1)} = \mathcal{R}_{\infty M} \mathcal{C}_{N,M} e^{(l)} = \mathcal{R}_{\infty M} \mathcal{C}_{N,M} \mathcal{P}_{\infty M} e^{(l)}, \quad l \geq 0.
\]
Consider the case $l = 0$ and note that $e^{(0)}$ can be chosen arbitrarily. Then we see that the action of $C_{N,M}$ and $R_{\infty M}C_{N,M}P_{\infty M}$ is the same on all $e^{(0)} \in \mathbb{R}^M$, and thus

(A.20) $$C_{N,M} = R_{\infty M}C_{N,M}P_{\infty M}.$$ 

To show (A.1), use

(A.21) $$\|C_{N,M}\| \leq \|R_{\infty M}C_{N,M}P_{\infty M}\| \leq \frac{5}{3} \|C_{N,M}\|.$$

The approximate surface case. When the surface $S$ is replaced by $\tilde{S}$, then the matrix elements in (A.10) over $S$ are replaced by the corresponding integrals over $\tilde{S}$. Let $\tilde{C}_{N,M}$ denote the matrix in (A.17) when using $\tilde{S}$ (actually using $\tilde{S}_N$ when defining $K_N$ and using $\tilde{S}_M$ when defining $K_M$). The result (A.1) is shown by a perturbation argument using

(A.22) $$\|C_{N,M} - \tilde{C}_{N,M}\| = O(h_N).$$

The matrix norm used in (A.21) is the row-norm induced by $\| \cdot \|_\infty$ on $\mathbb{R}^M$. We compare the integrals

(A.23) $$\int_{\sigma} K(v_j^{(M)}, m_k^{(M)}(s,t)) \ell_j(s,t) \left| (D_s m_k^{(M)} \times D_t m_k^{(M)}) (s,t) \right| |d\sigma|$$

based on the exact surface $S$, with

(A.24) $$\int_{\sigma} K(v_j^{(M)}, \tilde{m}_k^{(M)}(s,t)) \ell_j(s,t) \left| (D_s \tilde{m}_k^{(M)} \times D_t \tilde{m}_k^{(M)}) (s,t) \right| |d\sigma|,$$

which are based on the approximate surface $\tilde{S}$; we consider the corresponding integrals for the coarse mesh parameter $N$. The kernel function is, of course, the double layer function

$$K(P, Q) = \frac{\partial}{\partial v_Q} \left[ \frac{1}{|P - Q|} \right].$$

The measurement of the error in (A.24) and its effect on the row norm of $K_M$ and $K_N$ is based on a slight modification of a derivation given in the thesis of Chien [14]. Using this thesis, it can be shown that

(A.25) $$\|K_M - \tilde{K}_M\| = O(h_M), \quad \|K_N - \tilde{K}_N\| = O(h_N).$$

Chien [14, §3.6] looked at the approximation of the solid angle on the surface of $S$ at the nodes of $\mathcal{V}_M$. As this is a very involved derivation, based on asymptotic expansions of the error in powers of $h$, we omit it here. Using (A.25), the assertion (A.22) follows easily and also the assertion (A.1).

REFERENCES


