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The discrete collocation method for nonlinear integral equations

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The collocation method for solving linear and nonlinear integral equations results in many integrals which must be evaluated numerically. In this paper, we give a general framework for *discrete collocation methods*, in which all integrals are replaced by numerical integrals. In some cases, the collocation method leads to solutions which are superconvergent at the collocation node points. We consider generalizations of these results, to obtain similar results for discrete collocation solutions. Lastly, we consider a variant due to Kumar and Sloan for the collocation solution of Hammerstein integral equations.

1. Introduction

This paper presents and discusses a framework for analyzing *discrete collocation methods* for solving nonlinear integral equations. As a prototype equation, consider the Urysohn integral equation

$$x(s) = \int_D K(s, t, x(t))\,\mathrm{d}t \qquad s \in D \tag{1.1}$$

In this equation, D is a closed bounded set in $\mathbb{R}^m$, some $m \geq 1$, and the function $K(s, t, u)$ is such that the nonlinear integral operator in (1.1) is a completely continuous operator on some open domain $\Omega \subset C(D)$ into $C(D)$. The framework presented here will apply to more general nonlinear integral equations, but our work here will be for equations of the form (1.1).

Collocation is a popular numerical method for solving nonlinear integral equations, and there is a relatively large literature on its error analysis and implementation. For example, see Atkinson (1973), Kaneko, Noren, and Xu (1990), Krasnoselskii (1964), Krasnoselskii *et al.* (1972), Kumar (1987), (1988), Kumar and Sloan (1987), Moore (1966), (1968), Vainikko and Karma (1974), and Weiss (1974). The principal difficulty with this theory is that there are integrals which must usually be evaluated numerically, resulting in what we call the *discrete collocation method.* A few error analyses have been given for discrete collocation methods, e.g. see Ganesh and Joshi (1989) and Kumar (1988); but these have generally treated the discrete collocation method as a perturbation of the collocation method, or they have restricted their interest to particular collocation methods. In this paper, we give a direct and general treatment of the discrete collocation method, obtaining stronger results than seem possible with earlier types of analyses.

In Section 2, we review the collocation and discrete collocation method for linear integral equations. The framework for the discrete collocation method for linear integral equations is based on results from Flores (1990); but a similar framework was also given in Golberg (1990). Section 3 discusses the collocation and discrete collocation methods for nonlinear integral equations; and Section 4 gives some results on superconvergence at the collocation nodes of the discrete collocation solution. Section 5 considers the special case of Hammerstein nonlinear integral equations.

2. The discrete collocation method: linear equations

Consider the linear integral equation

$$x(s) - \int_D K(s, t)x(t)\,dt = y(s) \qquad s \in D \tag{2.1}$$

We write this as

$$x - \mathcal{K}x = y, \qquad y \in C(D) \tag{2.2}$$

with $C(D)$ the Banach space of continuous functions on D, with the uniform norm. The integral operator is denoted by $\mathcal{K}$, and it is a compact linear operator from $C(D)$ into $C(D)$. We assume that 1 is not an eigenvalue of $\mathcal{K}$, so that $I - \mathcal{K}$ has a bounded inverse on $C(D)$.

Let $\mathcal{X}_n$ be a finite dimensional subspace of $C(D)$ with basis $\{\phi_{1,n}, \ldots, \phi_{d,n}\}$, where $d \equiv d_n$ is the dimension of $\mathcal{X}_n$. Let $\{t_{1,n}, \ldots, t_{d,n}\}$ be a set of distinct *collocation node points* from D, and assume

$$\det [\phi_j(t_i)] \neq 0 \tag{2.3}$$

For given $x \in C(D)$, define

$$P_n x(s) = \sum_{j=1}^{d} \gamma_j \phi_j(s) \tag{2.4}$$

with $\gamma = [\gamma_1, \ldots, \gamma_d]^T$ chosen by solving the linear system

$$[\phi_j(t_i)]\gamma = [x(t_1), \ldots, x(t_d)]^T$$

In operator notation, the collocation method for solving (2.1) amounts to solving

$$x_n - P_n\mathcal{K}x_n = P_n y \tag{2.5}$$

for n sufficiently large.

The general theory of projection methods applies to collocation methods, and we refer the reader to Atkinson (1976, pp. 54–58). If we assume

$$\operatorname*{Limit}_{n\to\infty} \|x - P_n x\|_\infty = 0, \quad \text{all} \quad x \in C(D) \tag{2.6}$$

then it can be shown that

$$\|(I - P_n)\mathcal{K}\| \to 0 \quad \text{as} \quad n \to \infty \tag{2.7}$$

This is discussed further in Atkinson (1976, pp. 53–54). Using (2.7), it can then be shown that for sufficiently large n, say $n \geq N$, $(I - P_n\mathcal{K})^{-1}$ exists and is uniformly bounded,

$$\|(I - P_n\mathcal{K})^{-1}\| \leq M \equiv c\, \|(I - \mathcal{K})^{-1}\|, \qquad n \geq N \tag{2.8}$$

with c a suitable constant. [We will use c as a generic constant throughout this paper.] In addition, for suitable positive constants c_1 and c_2,

$$c_1\, \|x - P_n x\|_\infty \leq \|x - x_n\|_\infty \leq c_2\, \|x - P_n x\|_\infty, \qquad n \geq N \tag{2.9}$$

This says that the order of convergence of x_n to x is exactly that of the interpolation error $x - P_n x$.

An important variant on the collocation method (and more generally, on projection methods) is the *Sloan iterate* or *iterated collocation solution*:

$$\hat{x}_n = y + \mathcal{K}x_n \tag{2.10}$$

By applying P_n to both sides and using (2.5),

$$P_n\hat{x}_n = x_n \tag{2.11}$$

Substituting into (2.10), $\hat{x}_n$ satisfies the equation

$$\hat{x}_n - \mathcal{K}P_n\hat{x}_n = y \tag{2.12}$$

One of the main reasons for being interested in the iterated collocation solution is that it often converges more rapidly than does the original solution x_n. In particular,

$$\|x - \hat{x}_n\|_\infty \leq c\, \|\mathcal{K}(I - P_n)x\|_\infty \qquad n \geq N \tag{2.13}$$

and in many cases, the right side converges to zero more rapidly than does $x - P_n x$.

The discrete collocation method: linear equations To be more precise in our discussion of the further discretization of the collocation method, we set up the linear system that is used in solving the collocation equation (2.5). Since $x_n \in \mathcal{X}_n$, write

$$x_n(s) = \sum_{j=1}^{d} a_j \phi_j(s) \tag{2.14}$$

The coefficients $\{a_j\}$ are obtained by solving

$$\sum_{j=1}^{d} a_j \left[\phi_j(t_i) - \int_D K(t_i, t)\phi_j(t)\, \mathrm{d}t \right] = y(t_i) \qquad i = 1, \ldots, d_n \tag{2.15}$$

The integrals in (2.15) must usually be evaluated numerically.

To approximate these integrals, we introduce a numerical integration scheme

$$\mathcal{K}x(s) \equiv \int_D K(s, t)x(t)\, \mathrm{d}t \approx \mathcal{K}_n x(s) \equiv \sum_{k=1}^{R} w_k(s)x(\tau_k) \qquad s \in D \tag{2.16}$$

with arbitrary $x \in C(D)$. The integration nodes are $\{\tau_{1,n}, \ldots, \tau_{R,n}\}$ with $R \equiv R_n$ the number of integration node points associated with the index n. The weights $\{w_k(s)\}$ are allowed to be general enough to include integration methods based on product integration. It is assumed that

$$\operatorname*{Limit}_{n\to\infty} \|\mathcal{K}x - \mathcal{K}_n x\|_\infty = 0, \quad \text{all} \quad x \in C(D) \tag{2.17}$$

Further assumptions on $\{\mathcal{K}_n\}$ are given later.

The *discrete collocation method* is defined by the approximating equation

$$z_n - P_n\mathcal{K}_n z_n = P_n y \tag{2.18}$$

and the *iterated discrete collocation solution* is defined by

$$\hat{z}_n = y + \mathcal{K}_n z_n \tag{2.19}$$

In analogy with the collocation method,

$$P_n\hat{z}_n = z_n \tag{2.20}$$

and $\hat{z}_n$ satisfies

$$\hat{z}_n - \mathcal{K}_n P_n \hat{z}_n = y \tag{2.21}$$

To see more precisely the effect of using (2.18), we let

$$z_n(s) = \sum_{j=1}^{d} \beta_j \phi_j(s) \tag{2.22}$$

The coefficients $\{\beta_j\}$ are then obtained from solving

$$\sum_{j=1}^{d} \beta_j\left[\phi_j(t_i) - \sum_{k=1}^{R} w_k(t_i)\phi_j(\tau_k)\right] = y(t_i) \qquad i = 1, \ldots, d_n \tag{2.23}$$

This corresponds to the numerical integration of the integrals in (2.15) by using the method in (2.16).

The standard means of analyzing the discrete collocation method is to regard (2.23) as a perturbation of the earlier system (2.15); and then the solvability theory for (2.15) is combined with perturbation theory to give an error analysis for (2.23), and thence for (2.18). For example, see Joe (1985, p. 1172), in which an error analysis is given for collocation with piecewise polynomial functions. In contrast, the present framework gives a direct analysis of (2.18) and (2.21), one not using the convergence results for the collocation equations (2.5) and (2.12). The present approach was first given in Flores (1990) and Golberg (1990, pp. 113–115). We begin with the special case in which $d_n = R_n$.

THEOREM 1 Assume $d_n = R_n$, and further assume that the collocation nodes $\{t_i\}$ and the integration nodes $\{\tau_i\}$ are the same. Assume $(I - \mathcal{K}_n P_n)^{-1}$ exists. Then the iterated discrete collocation solution $\hat{z}_n$ is the exact solution to the Nyström equation

$$z - \mathcal{K}_n z = y \tag{2.24}$$

Proof. It is sufficient to note that

$$\mathcal{K}_n P_n z = \mathcal{K}_n z$$

for all $z \in C(D)$, from the above hypotheses. □

As a consequence of this theorem, the convergence of a number of well-known cases follows from the error analysis for the Nyström method. As an example of such a result, see Joe (1985, p. 1174). One of the main results in this latter paper analyzes the case with $\mathscr{X}_n$ defined as piecewise polynomial functions that are piecewise continuous, with the numerical integration a composite Gauss–Legendre quadrature. [For an analysis of the Nyström method, see Anselone (1971) or Atkinson (1976).]

We consider now the more general case, in which $R_n \neq d_n$. The solvability of (2.18) and (2.21) are closely related, as is also true of (2.5) and (2.12) for the collocation method. If $(I - P_n\mathcal{K}_n)^{-1}$ exists, then

$$(I - \mathcal{K}_n P_n)^{-1} = I + \mathcal{K}_n(I - P_n\mathcal{K}_n)^{-1}P_n \tag{2.25}$$

Conversely, if $(I - \mathcal{K}_n P_n)^{-1}$ exists, then

$$(I - P_n\mathcal{K}_n)^{-1} = I + P_n(I - \mathcal{K}_n P_n)^{-1}\mathcal{K}_n \tag{2.26}$$

With these results, we are free to choose to analyze either the discrete collocation method or the iterated discrete collocation method, whichever is more convenient; and we choose the latter.

THEOREM 2 Let the integral equation $(I - \mathcal{K})x = y$ be uniquely solvable for $y \in C(D)$. Assume $\{\mathcal{K}_n\}$ is a collectively compact family of the approximations to $\mathcal{K}$, and assume it satisfies the pointwise convergence of (2.17). Assume $\{P_n\}$ is a uniformly bounded and pointwise convergent family of projection operators on $C(D)$. Then for all sufficiently large n, say $n \geq N$, $(I - \mathcal{K}_n P_n)^{-1}$ exists and is uniformly bounded. Moreover, if $\hat{z}_n$ is the solution to (2.21), then

$$\|x - \hat{z}_n\|_\infty \leq \|(I - \mathcal{K}_n P_n)^{-1}\| \, \|\mathcal{K}x - \mathcal{K}_n P_n x\|_\infty, \qquad n \geq N \tag{2.27}$$

Proof. It is straightforward to show $\{\mathcal{K}_n P_n\}$ is a collectively compact and pointwise convergent family on $C(D)$; and then the theorem follows from known results, e.g. Anselone (1971). □

3. The discrete collocation method: nonlinear equations

The nonlinear integral equation is written abstractly as

$$x = \mathcal{K}(x) \tag{3.1}$$

It is assumed that $\mathcal{K} : \Omega \subset C(D) \to C(D)$ is a completely continuous operator with Ω an open set. We will always take $\mathscr{X} = C(D)$, although this could be generalized to subspaces of $C(D)$ with norms that imply uniform convergence. With the notation of Section 2, the collocation method for (3.1) is defined as

$$x_n = P_n \mathcal{K}(x_n) \tag{3.2}$$

The iterated collocation method is defined by

$$\hat{x}_n = \mathcal{K}(x_n) \tag{3.3}$$

The first general convergence results for the collocation method (3.2) seem to have been given by M. A. Krasnoselskii around 1950 (see Krasnoselskii (1964)). The analysis of the approximating equation was carried out using the Schauder–Leray degree theory for completely continuous vector fields. Other analyses of the collocation method (3.2) have since been given, including Kumar (1987), Kumar–Sloan (1987), and Weiss (1974). For an analysis of the iterated collocation method, see Atkinson–Potra (1987).

As with the linear case, let $\mathcal{K}(x)$ be approximated by a 'numerical integration operator',

$$\mathcal{K}(x) \approx \mathcal{K}_n(x) \quad \text{for all } x \in \Omega \tag{3.4}$$

For the Urysohn integral operator of (1.1), this would usually take the form of a standard numerical integration. For example,

$$\int_D K(s, t, x(t))\, dt \approx \sum_{j=1}^{R} w_j K(s, \tau_j, x(\tau_j)), \qquad x \in \Omega \tag{3.5}$$

with $R \equiv R_n$, as in Section 2. More general forms are possible, for example, to compensate for singular integrands by using product integration. But it is always assumed that $\mathcal{K}_n(x)$ uses only the values $\{x(\tau_1), \ldots, x(\tau_R)\}$ in approximating $\mathcal{K}(x)$ to obtain $\mathcal{K}_n(x)$.

To see more explicitly the form of the collocation system that must be approximated, let

$$x_n(s) = \sum_{j=1}^{d} a_j \phi_j(s)$$

Substituting into the Urysohn equation (1.1) and collocating at the node points $\{t_i, 1 \leq i \leq d_n\}$ yields

$$\sum_{j=1}^{d} a_j \phi_j(t_i) = \int_D K\left(t_i, t, \sum_{j=1}^{d} a_j \phi_j(t)\right) dt, \qquad i = 1, \ldots, d_n \tag{3.6}$$

The discrete collocation method replaces the integral with a numerical integral, such as that in (3.5).

The discrete collocation method for solving (3.1) is defined by

$$z_n = P_n \mathcal{K}_n(z_n) \tag{3.7}$$

The iterated discrete collocation solution is defined by

$$\hat{z}_n = \mathcal{K}_n(z_n) \tag{3.8}$$

As in (2.20),

$$z_n = P_n \hat{z}_n \tag{3.9}$$

and thus z_n and $\hat{z}_n$ agree at the collocation points $\{t_i\}$. Using (3.9) in (3.8), we have that $\hat{z}_n$ satisfies

$$\hat{z}_n = \mathcal{K}_n(P_n\hat{z}_n) \tag{3.10}$$

We give an analysis first for the iterated discrete collocation solution, and then the convergence of $\{z_n\}$ will follow from

$$x - z_n = [x - P_n x] + P_n[x - \hat{z}_n] \tag{3.11}$$

Also, from (3.9)

$$\underset{1 \le i \le d}{\text{Maximum}}\, |x(t_i) - z_n(t_i)| \le \|x - \hat{z}_n\|_\infty \tag{3.12}$$

The convergence analysis for (3.10) is based on Atkinson (1973); but Weiss (1974) could also have been used. Following are hypotheses for the approximating operators $\{\mathcal{K}_n \mid n \ge 1\}$. These are satisfied by essentially all numerical integration operators used in practice.

H1. $\mathcal{K}_n : \Omega \subset C(D) \to C(D)$ are completely continuous operators. In addition, $\mathcal{K}_n(x)$ depends on x at only the node points $\{\tau_i \mid 1 \le i \le R_n\}$.

H2. $\{\mathcal{K}_n\}$ is a collectively compact family on D: for every bounded set $B \subset \Omega$, the set $\{\mathcal{K}_n(B) \mid n \ge 1\}$ is precompact in $C(D)$.

H3. For every $x \in \Omega$, $\mathcal{K}_n(x) \to \mathcal{K}(x)$ as $n \to \infty$.

H4. At each $x \in \Omega$, $\{\mathcal{K}_n\}$ is an equicontinuous family.

H5. For a given fixed point x_* of $\mathcal{K}$ and $r > 0$, assume $\mathcal{K}$ and $\mathcal{K}_n$, $n \ge 1$, are twice Frechet differentiable on $B(x_*, \varepsilon_0) \equiv \{x \mid \|x - x_*\| \le \varepsilon_0\}$ with

$$\|\mathcal{K}^*(x)\|,\ \|\mathcal{K}_n^*(x)\| \le M, \qquad x \in B(x_*, \varepsilon_0), \quad n \ge 1,$$

with M a positive constant.

THEOREM 3 Assume $\mathcal{K} : \Omega \subset C(D) \to C(D)$ is a completely continuous operator with Ω an open set; and in addition, let x_* be an isolated fixed point of $\mathcal{K}$ of nonzero (Schauder–Leray) index. Assume $\{\mathcal{K}_n \mid n \ge 1\}$ satisfies the above hypotheses H1–H4. Finally, assume that $d_n = R_n$ and $\{t_{i,n}\} = \{\tau_{i,n}\}$, for all $n \ge 1$. Then

(a) The iterated discrete collocation method coincides with the Nyström method $z = \mathcal{K}_n(z)$. From this it follows that

(b) there is a radius $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$, there is an $N_\varepsilon > 0$, for which $n \ge N_\varepsilon$ implies the function $\mathcal{K}_n$ has no fixed point in the neighbourhood $\{x \mid \varepsilon \le \|x - x_*\| \le \varepsilon_0\}$; and

(c) For any given radius $\varepsilon > 0$, there is an $N_\varepsilon > 0$, such that for all $n \ge N_\varepsilon$, the function $\mathcal{K}_n$ has at least one fixed point $\hat{z}_n$ within ε of x_*.

Consequently, the fixed points of the approximating equation approximate x_* within some fixed neighbourhood $\{x \mid \|x - x_*\| \le \varepsilon_0\}$ of x_*. If in addition, H5 is satisfied and 1 is not an eigenvalue of $\mathcal{K}'(x_*)$, then

(d) For ε_0 sufficiently small, and for N sufficiently large, the fixed points $\hat{z}_n$ of $\mathcal{K}_n$ within $B(x_*, \varepsilon_0)$ exist and are unique, for $n \ge N$; and for some constant $c > 0$,

$$\|x_* - \hat{z}_n\| \le c\, \|\mathcal{K}(x_*) - \mathcal{K}_n(x_*)\|, \qquad n \ge N \tag{3.13}$$

Proof. For any x, $\mathscr{K}_n(P_n x) = \mathscr{K}_n(x)$. This follows from H1 and the definition of $P_n x$, that

$$P_n x(t_i) = x(t_i), \qquad i = 1, \ldots, d_n$$

The remainder of the theorem is then a simple restatement of Theorems 3 and 4 of Atkinson (1973). □

This is an important result, since in much of the literature for discrete collocation methods for nonlinear equations, $d_n = R_n$ and $\{t_{i,n}\} = \{\tau_{i,n}\}$. By a direct analysis of the right side of (3.13), improved error results can be obtained, as in Atkinson (1973) and Weiss (1974).

For the discrete collocation method with $d_n \neq R_n$, the result is only slightly more complex.

THEOREM 4 Assume the hypotheses of Theorem 3, including H1–H4. In addition, assume (2.6), that $\{P_n\}$ is pointwise convergent to I on $C(D)$; and assume that $P_n(\Omega) \subset \Omega$, $n \geq 1$. Then the conclusions (b) and (c) of Theorem 3 are still valid. If H5 is also assumed, then (c) of Theorem 3 is also valid, with

$$\|x_* - \hat{z}_n\| \leq c\,\|\mathscr{K}(x_*) - \mathscr{K}_n(P_n x_*)\|, \qquad n \geq N \tag{3.14}$$

Proof. First show that the family $\{\hat{\mathscr{K}}_n \mid n \geq 1\}$

$$\hat{\mathscr{K}}_n(x) = \mathscr{K}_n(P_n x), \qquad x \in \Omega,$$

satisfies the hypotheses H1–H4 given above, with $\mathscr{K}_n$ replaced by $\hat{\mathscr{K}}_n$; and then the remainder of the proof follows from Atkinson (1973). As the proof H1–H4 for $\{\hat{\mathscr{K}}_n\}$ is straightforward, we omit it. □

4. Applications and superconvergence

It is well-known that for some choices of approximating subspaces $\mathscr{X}_n$ and interpolation nodes $\{t_{in}\}$, $n \geq 1$, the collocation solution x_n is superconvergent at the node points. To be more precise, let

$$E_n = \operatorname*{Max}_{1 \leq i \leq d_n} |x(t_i) - x_n(t_i)| \tag{4.1}$$

Then superconvergence at the nodes occurs if

$$\operatorname*{Limit}_{n \to \infty} \frac{E_n}{\|x - x_n\|_\infty} = 0 \tag{4.2}$$

We wish to consider whether superconvergence results for collocation methods will extend to similar results for discrete collocation methods.

We begin by considering numerical methods for the linear integral equation

$$x(s) = \int_a^b K(s, t)x(t)\,\mathrm{d}t = y(s), \qquad a \leq s \leq b \tag{4.3}$$

with the standard assumptions that $\mathscr{X} = C[a, b]$ and the integral operator $\mathscr{K}$ is compact on $\mathscr{X}$ into $\mathscr{X}$. For the numerical method, let $r > 0$, $n > 0$, $h = (b-a)/n$,

$T_k = a + kh$ for $k = 0, \ldots, n$, and

$$\mathscr{X}_n = \{g \in L^\infty(a, b) \mid g(t) \text{ is a polynomial of degree} < r \text{ on each subinterval } (T_{k-1}, T_k),\ k = 1, \ldots, n\} \tag{4.4}$$

Let

$$0 < q_1 < \cdots < q_r < 1 \tag{4.5}$$

denote the Gauss–Legendre zeros of order r on $[0, 1]$, and define the collocation nodes by

$$t_{ki} = T_{k-1} + q_i h, \qquad i = 1, \ldots, r, \quad k = 1, \ldots, n \tag{4.6}$$

Assuming sufficient differentiability for x and $K(s, t)$,

$$\|x - x_n\|_\infty = O(h^r), \qquad \|x - \hat{x}_n\|_\infty = O(h^{2r}) \tag{4.7}$$

For the linear case, see Chatelin and Lebbar (1984); and for the nonlinear case, see Atkinson and Potra (1987). We show that this carries over to suitably defined discrete collocation methods.

Using the error formula (2.27) for the iterated discrete collocation method for the linear case, we analyze the error $\|\mathscr{K}x - \mathscr{K}_n P_n x\|_\infty$; and we use the decomposition

$$\|\mathscr{K}x - \mathscr{K}_n P_n x\|_\infty \le \|\mathscr{K}x - \mathscr{K}_n x\|_\infty + \|\mathscr{K}_n(x - P_n x)\|_\infty \tag{4.8}$$

The integration scheme should be so chosen that

$$\|\mathscr{K}x - \mathscr{K}_n x\|_\infty = O(h^{2r}) \tag{4.9}$$

in agreement with the error for $\hat{x}_n$ in (4.7). This will be true with a composite integration rule based on a standard quadrature method applied to each subinterval $[T_{k-1}, T_k]$ and having degree of precision of $2r - 1$ or larger.

Next we want to have the basic quadrature rule used in defining the composite method integrate exactly the identities

$$\int_0^1 t^l (t - q_1) \cdots (t - q_r)\, dt = 0, \qquad 0 \le l \le r - 1 \tag{4.10}$$

for reasons given below. The result (4.10) also follows from the quadrature rule having a degree of precision of at least $2r - 1$. Using (4.10), we give an analysis of the error $\|\mathscr{K}_n(x - P_n x)\|_\infty$.

Using a composite integration rule with ρ quadrature nodes per subinterval $[T_{k-1}, T_k]$, define

$$\mathscr{K}_n z(s) = h \sum_{k=1}^{n} \sum_{i=1}^{\rho} w_i K(s, \tau_{ki}) z(\tau_{ki}) \qquad z \in C[a, b] \tag{4.11}$$

If this has a degree of precision of at least $2r - 1$, and if $K(s, t)$ is $2r$-times continuously differentiable, then (4.9) follows easily.

For the final term $\|\mathscr{K}_n(x - P_n x)\|_\infty$ in (4.8), write

$$\mathscr{K}_n(I - P_n)x(s) = h \sum_{k=1}^{n} \sum_{i=1}^{\rho} w_i K(s, \tau_{ki})[x(\tau_{ki}) - P_n x(\tau_{ki})] \tag{4.12}$$

Recall the following standard error formula for polynomial interpolation of degree r. For $\tau \in [T_{k-1}, T_k]$:

$$x(\tau) - P_n x(\tau) = h^r \psi((\tau - T_{k-1})/h) x[t_{k1}, \ldots, t_{kr}, \tau] \tag{4.13}$$

using the Newton divided difference of x of order r and

$$\psi(\tau) \equiv (\tau - q_1) \cdots (\tau - q_r) \tag{4.14}$$

Define

$$g_{s,k}(t) = K(s, t) x[t_{k1}, \ldots, t_{kr}, t]$$

for $T_{k-1} < t < T_k$, $a \leq s \leq b$. Using (4.12),

$$\mathcal{K}_n(I - P_n)x(s) = r + 1 \sum_{k=1}^{n} \sum_{i=1}^{\rho} w_i g_{s,k}(\tau_{ki}) \psi((\tau_{ki} - T_{k-1})/h) \tag{4.15}$$

If $x \in C^{2r}[a, b]$ and $K(s, t)$ is r-times continuously differentiable with respect to t, uniformly in s, then we can expand $g_{s,k}(\tau)$ about T_{k-1}:

$$g_{s,k}(\tau) = \sum_{l=0}^{r-1} \frac{1}{l!} (\tau - T_{k-1})^l g_{s,k}^{(l)}(T_{k-1}) + O(h^r) \tag{4.16}$$

Using (4.10), it follows that

$$\|\mathcal{K}_n(x - P_n x)\|_\infty = O(h^{2r}) \tag{4.17}$$

Recalling (2.27), and using (4.8), (4.9), and (4.17), we complete the proof that

$$\|x - \hat{z}_n\|_\infty = O(h^{2r}) \tag{4.18}$$

Note that in the case that $\mathcal{K}_n$ in (4.11) is defined by using Gauss–Legendre quadrature with $\rho = r$, we have that $\mathcal{K}_n P_n = \mathcal{K}_n$ and Theorem 1 applies. Thus there is no real need to use a more accurate integration formula, although that is done in some cases.

To generalize these results to the nonlinear Urysohn integral equation

$$x(s) = \int_a^b K(s, t, x(t))\, dt, \quad a \leq s \leq b \tag{4.19}$$

consider the error formula (3.14). Write

$$\|\mathcal{K}(x_*) - \mathcal{K}_n(P_n x_*)\|_\infty \leq \|\mathcal{K}(x_*) - \mathcal{K}_n(x_*)\|_\infty + \|\mathcal{K}_n(x_*) - \mathcal{K}_n(P_n x_*)\|_\infty \tag{4.20}$$

The term $\|\mathcal{K}(x_*) - \mathcal{K}_n(x_*)\|_\infty$ is treated in much the same way as for the linear case. For the final term in (4.19), write

$$\mathcal{K}_n(x_*) - \mathcal{K}_n(P_n x_*) = \mathcal{K}_n'(x_*)(x_* - P_n x_*) + O(\|x_* - P_n x_*\|^2) \tag{4.21}$$

The last term is known and leads to an error of size $O(h^{2r})$ for $x \in C^r[a, b]$.

For the first term on the right side of (4.21), (3.5) yields

$$\mathcal{K}_n'(x_*)(x_* - P_n x_*)(s) = \sum_{i=1}^{R} w_i K_u(s, \tau_i, x_*(\tau_i))[x_*(\tau_i) - P_n x_*(\tau_i)] \tag{4.22}$$

This can now be treated in the same way as that of the linear integral operator term in (4.17). In this way, the convergence results for the iterated discrete collocation solution $\hat{z}_n$ and the superconvergence results for z_n generalize easily from the case of linear integral equations to the nonlinear Urysohn equation (1.1).

The above derivation extends to other well-known collocation methods. For example, the use of continuous piecewise quadratic interpolation to define P_n leads to

$$\|x - x_n\|_\infty = O(h^3), \qquad \|x - \hat{x}_n\|_\infty = O(h^4) \tag{4.23}$$

If the quadrature method for the approximating operator $\mathcal{K}_n$ is defined with a composite rule with degree of precision three on each subinterval $[T_{k-1}, T_k]$, then the results in (4.23) carry across to the discrete solutions z_n and $\hat{z}_n$.

Multivariable integral equations. Consider the solution of multivariable integral equations (1.1) and (2.1) with $D \subset \mathbb{R}^m$, $m > 1$. The theory of collocation methods for such equations is not as well developed as for the one variable case with $D = [a, b]$; and very few superconvergence results are known. Nonetheless, the theory of multivariable interpolation is well-developed, and we can then say something about the order needed for the numerical integration scheme in order to preserve the order of convergence of $\|x_* - x_n\|_\infty$. As a particular example, we consider the solution of integral equations over piecewise smooth surfaces in $\mathbb{R}^3$.

Let D be a surface in $\mathbb{R}^3$, which we decompose into smooth closed subsurfaces:

$$D = D_1 \cup \cdots \cup D_J \tag{4.24}$$

If distinct subsurfaces D_i and D_j intersect, then the intersection is to be a portion of the boundary of each one. Each D_i is to be a smooth surface; or more precisely, assume the existence of a smooth mapping

$$F_i : \hat{D}_i \xrightarrow[\text{onto}]{1-1} D_i \qquad i = 1, \ldots, J \tag{4.25}$$

with $\hat{D}_i$ a polygonal region in the plane $\mathbb{R}^2$. Later we are more precise about the needed smoothness for each F_i.

Let $\{\hat{\Delta}_{i,k}\}$ be a triangulation of $\hat{D}_i$, and let $\Delta_{i,k} = F_i(\hat{\Delta}_{i,k})$ define a corresponding triangulation of D_i. Collecting together these triangulations of each D_i, we have a triangulation $\mathcal{T}_n = \{\Delta_1^{(n)}, \ldots, \Delta_n^{(n)}\}$ of D; and we usually dispense with the superscript n, understanding it implicitly. We further assume a compatibility of neighbouring triangles in D: if $\Delta_k^{(n)}$ and $\Delta_l^{(n)}$ come from distinct subsurfaces D_i and D_j, respectively, and if $\Delta_k^{(n)}$ and $\Delta_l^{(n)}$ have a nonempty intersection, then the intersection is either (1) a single point, consisting of a vertex of both triangles, or (2) a common edge of both triangles, lying in the common portion of the boundary of D_i and D_j. In addition, the union of the vertices of the triangles in $\mathcal{T}_n$ is to contain all vertices of the original surface D; and the union of the edges of the triangles in $\mathcal{T}_n$ is to contain all edges of D. For an additional discussion of the triangulation of D, see Atkinson (1985a), (1985b).

Let $r \geq 1$, and consider a polynomial $\hat{p}(\hat{x}, \hat{y})$ of degree $\leq r$ over some triangular element $\hat{\Delta}_{i,k}$. Then define a corresponding function p over $\Delta_{i,k}$ by

$$p(F_i(\hat{x}, \hat{y})) = \hat{p}(\hat{x}, \hat{y}), \qquad (\hat{x}, \hat{y}) \in \hat{\Delta}_{i,k} \tag{4.26}$$

Given a piecewise polynomial function $\hat{p}$ (not necessarily continuous) of degree $\leq r$ over $\hat{D}_i$, with $\hat{p}$ polynomial over each $\hat{\Delta}_{i,k}$, we can define a corresponding function p over D_i by using (4.26); and given such 'piecewise polynomial' functions p over each D_i, we can define a corresponding piecewise polynomial function p over D. Let $\bar{\mathscr{X}}_n$ denote the collection of all such piecewise polynomial functions over D of degree $\leq r$. If there are no continuity requirements, then the dimension of $\bar{\mathscr{X}}_n$ is

$$d_n = n(r+1)(r+2)/2 \tag{4.27}$$

With a triangulation of D and the use of piecewise polynomial functions such as the above, the simplest way to define collocation is to use a uniform subdivision of each triangle $\hat{\Delta}_{i,k}$. To introduce the collocation points, first consider the unit simplex $\sigma = \{(s, t) \mid 0 \leq s, t, s+t \leq 1\}$ in the plane. Let $\delta = 1/r$, and define a grid for σ by

$$T = \{(s_i, t_j) = (i\delta, j\delta) \mid 0 \leq s_i, t_j, s_i + t_j \leq 1\} \tag{4.28}$$

For a triangle $\hat{\Delta} \subset \mathbb{R}^2$, let $\{v_1, v_2, v_3\}$ denote its vertices; and then define $m : \sigma \xrightarrow[\text{onto}]{1-1} \hat{\Delta}$ by

$$m(s, t) = uv_1 + tv_2 + sv_3 \qquad u = 1 - s - t \tag{4.29}$$

For collocation nodes in such a $\hat{\Delta}$, we choose the points $\{m(q_k) \mid q_k \in T\}$. We determine nodes over each D_i in this manner, and thus over D as well. Denote these node points by $\{\tau_k \mid 1 \leq k \leq d_n\}$. Because these nodes are shared between neighbouring triangles, the definition of the interpolation subspace should be modified. We use

$$\mathscr{X}_n = \bar{\mathscr{X}}_n \cap C(D)$$

For a simple closed surface D, $d_n = \frac{1}{2}r^2 n + 2$.

With this choice of collocation nodes, the interpolation projection $P_n x \in C(D)$. For the error in the interpolation function $P_n x$, first assume each mapping function $F_i \in C^{r+1}(\hat{D}_i)$. Further assume $x \in C(D)$ and $x \mid D_i \in C^{r+1}(D_i)$, $i = 1, \ldots, J$. Then

$$\|x - P_n x\|_\infty = O(h^{r+1}) \tag{4.30}$$

where

$$h = \underset{\Delta_i \in \mathscr{T}_n}{\text{Max}} \operatorname{diam}(\hat{\Delta}_i)$$

where $\hat{\Delta}_i$ is the planar triangle corresponding to Δ_i on D. This error formula is a well-known result; for example, see Atkinson (1985a), where a proof is sketched for the case $r = 2$.

We now consider the collocation method for solving both the linear integral equation (2.1) and the nonlinear integral equation (1.1). Use the preceding definition of $\mathscr{X}_n$ and nodes $\{\tau_i\}$. With standard solvability assumptions for the integral equation (as in Theorems 1 and 3), we obtain

$$\|x_* - x_n\|_\infty = O(\|x_* - P_n x_*\|_\infty) \tag{4.31}$$

with x_* the unknown solution we are seeking and x_n the collocation solution. With sufficient differentiability for x_*, (4.30) implies

$$\|x_* - x_n\|_\infty = O(h^{r+1}) \tag{4.32}$$

For additional details in the case $r=2$, see Atkinson (1985a, 1985b).

The only superconvergence results of which we know are in Chien (1991), for the case $r=2$, If the triangulation is refined in a special way, then superconvergence at the nodes will result. For each $\Delta \in \mathscr{T}_n$, subdivide the corresponding planar triangle $\hat{\Delta}$ into four new triangles by connecting the midpoints of its sides. This causes the number of triangles to increase by a factor of four with each subdivision. With this method of refinement of the triangulation, Chien (1991) shows the convergence result

$$\operatorname*{Max}_i |x_*(\tau_i) - x_n(\tau_i)| = O(h^4) \tag{4.33}$$

Although the proof in Chien (1991) is given for only $r=2$, it appears to generalize to all even values of $r \geqslant 2$, and then the result is

$$\operatorname*{Max}_i |x_*(\tau_i) - x_n(\tau_i)| = O(h^{r+2}) \tag{4.34}$$

The results in Chien also include an analysis of the effects of the approximation of the surface D with the same form of interpolation as is used for the collocation the scheme; and he shows the order of convergence remains the same.

The numerical integration of the collocation integrals in the above method uses composite integration rules based on quadrature over planar triangles. In Table 1, we reference some basic integration formulas over triangles from the paper Lyness and Jespersen (1975), and we give some information about them. The

TABLE 1
Quadrature formulas over triangles

ρ	ν	$\bar{\nu}$	Type
2	3	1·5	[000100]
3	7	3	[111000]
4	9	5	[011100]
5	10	6	[111100]
6	13	10	[100210]
7	16	10·5	[110210]
8	16	16	[100301]
9	22	18	[111301]

degree of precision of the formula is denoted by ρ. The column labelled v gives the number of quadrature node points in the triangle; and $\bar{v}$ gives the average number of quadrature nodes per triangle when the rule is used in a composite quadrature formula. The column labelled *Type* is the identifier for the quadrature rule as given in Lyness and Jespersen (1975, 29–31).

Consider the numerical integration of the integral

$$\int_D g(Q)\,\mathrm{d}S = \sum_{l=1}^{n} \int_{\Delta_l} g(Q)\,\mathrm{d}S = \sum_{i=1}^{J} \sum_{k} \int_{\hat{\Delta}_{i,k}} g(F_i(x, y))\,|D_x F_i \times D_y F_i|\,\mathrm{d}x\,\mathrm{d}y \quad (4.35)$$

D_x and D_y denote the partial derivatives with respect to x and y, respectively. Apply a quadrature formula for triangles to the integrals on the right side, and assume the quadrature rule has degree of precision ρ. Then it can be shown that the error in the resulting quadrature is $O(h^{\rho+1})$, provided $g \mid D_i \in C^{\rho+1}(D_i)$ and $F_i \in C^{\rho+1}(\hat{D}_i)$, $i = 1, \ldots, J$. This can be improved to $O(h^{\rho+2})$ when ρ is even and the triangulation is refined in the manner described preceding (4.33); but we do not discuss the details of such in this paper.

Using the above quadrature, we can define a discrete collocation method for the collocation method described preceding (4.31). Theorems 2 and 4 imply that in order to preserve the order of convergence h^{r+1} in the collocation solution, we must choose a quadrature formula for triangles with $\rho = r$. An appropriate formula can be selected from Table 1.

The results of the last paragraph can be improved in some cases. Chien (1991) preserves the superconvergence result (4.33) for $r = 2$, and he uses the formula in Table 1 with $\rho = 2$, which has a lower degree of precision than one would think is necessary. Similar results are probably true for higher degree interpolation with even r.

5. Hammerstein integral equations

Consider the Hammerstein integral equation

$$x(t) = y(t) + \int_D L(t, s) g(s, x(s))\,\mathrm{d}s \qquad t \in D \quad (5.1)$$

This is written symbolically as

$$x = y + \mathcal{L}\mathcal{G}(x) \quad (5.2)$$

and to tie it in with earlier notation, we let $\mathcal{K}(x) \equiv y + \mathcal{L}\mathcal{G}(x)$. In this, we let $\mathcal{L}$ denote the linear integral operator with kernel function $L(t, s)$, and we assume $\mathcal{L}$ is compact on $C(D)$ to $C(D)$. We define $\mathcal{G}(x)(t) \equiv g(t, x(t))$, for all $x \in \Omega$, for some open domain $\Omega \subset C(D)$. In addition, we often assume

$$\mathcal{G} : \Omega \cap C^m(D) \to C^m(D) \quad (5.3)$$

for a given integer $m > 0$, to obtain differentiability results for $x(t)$.

In using the collocation and Galerkin methods to solve (5.1), one must usually solve a finite nonlinear system by some type of iteration method. In so doing, there is a need to do repeated numerical integrations involving the iterates, and this causes a high operations cost. To reduce this cost significantly, Kumar and Sloan (1987) introduced a variant procedure for solving (5.1).

In symbolic form, introduce

$$u = \mathscr{G}(x) \tag{5.4}$$

Then x and u satisfy

$$u = \mathscr{G}(y + \mathscr{L}u) \tag{5.5a}$$

$$x = y + \mathscr{L}u \tag{5.5b}$$

Galerkin and collocation methods can be applied to (5.5a), and this leads to a nonlinear system which is less costly to solve by iteration, because far fewer numerical integrations are needed.

Let P_n denote the interpolation projection associated with collocation. Equations (5.5) are approximated by

$$u_n = P_n\mathscr{G}(y + \mathscr{L}u_n) \tag{5.6a}$$

$$x_n = y + \mathscr{L}u_n \tag{5.6b}$$

The standard convergence analyses for the collocation method, of the type used in Theorems 1 and 2, can be applied to this approximation scheme.

We make the following assumptions regarding the equations (5.2) and (5.5). Let x_* denote an isolated fixed point of $\mathscr{K}$; and say it is isolated within $B(x_*, \varepsilon_0)$, the ball of radius ε_0 about x_*. Assuming $g(t, v)$ is differentiable, introduce the Frechet derivatives

$$\mathscr{G}'(x)w(t) \equiv \frac{\partial g(t, x(t))}{\partial v} w(t) \qquad t \in D, \quad w \in C(D)$$

$$\mathscr{G}''(x)(w, z)(t) \equiv \frac{\partial^2 g(t, x(t))}{\partial v^2} w(t)z(t) \qquad t \in D, \quad w, z \in C(D)$$

Assume $\mathscr{G}'(x)$ and $\mathscr{G}''(x)$ are bounded linear operators on $C(D)$ and $C(D) \times C(D)$, respectively, uniformly for $x \in B(x_*, \varepsilon_0)$.

Further assume that $[I - \mathscr{K}'(x_*)]^{-1}$, exists as a bounded operator on $C(D)$ to $C(D)$, with $\mathscr{K}'(x_*) = \mathscr{L}\mathscr{G}'(x_*)$. This implies that x_* has nonzero Schauder–Leray index. Moreover, it follows that (5.5a) has solution $u_*(t) = g(t, x_*(t))$ of nonzero index, with x_* isolated within some ball $B(u_*, \delta_0)$; and it can be shown that $[I - \mathscr{G}'(x_*)\mathscr{L}]^{-1}$ exists as a bounded operator on $C(D)$, thus implying u_* also has nonzero Schauder–Leray index.

We assume that the collocation projection operators P_n are pointwise convergent on $C(D)$, as in (2.8), although this can be weakened. Then the use of Krasnoselskii (1964) shows that for some $N > 0$, (5.6a) has a unique solution $u_n \in B(u_*, \delta_0)$, $n \geq N$, and $u_n \to u_*$ as $n \to \infty$. For the analysis of the rates of convergence, we can show that

$$\|u_* - u_n\|_\infty = O(\|u_* - P_n u_*\|_\infty) \tag{5.7}$$

with u_* the desired solution for (5.5a). This has been improved by Kumar and

Sloan, who show that

$$\|x_* - x_n\|_\infty \leq c\, \|\mathscr{L}(u_* - P_n u_*)\|_\infty + \mathrm{O}(\|u_* - P_n u_*\|_\infty^2) \tag{5.8}$$

where x_* is the desired solution of the original equation (5.2) and $u_* = \mathscr{G}(u_*)$. With many interpolation schemes, $\|\mathscr{L}(u_* - P_n u_*)\|_\infty$ can be shown to converge to zero more rapidly than $\|u_* - P_n u_*\|_\infty$, thus giving a faster rate of convergence than would be expected from the error result (5.7) for $u_* - u_n$. For example, the ideas of Chatelin and Lebbar (1984) would apply in this situation. For further discussion of this and discrete analogues, see Kumar (1987), (1988), Ganesh and Joshi (1989), (1991).

Discrete collocation for Hammerstein equations. With any scheme for discrete collocation, we want to be able to retain the various results described above for the collocation method. Define the numerical integration operation $\mathscr{L}_n$ by

$$\mathscr{L}_n u(t) = \sum_{k=1}^{R} w_k L(t, \tau_k) u(\tau_k) \qquad u \in C(D) \tag{5.9}$$

in analogy with the operator $\mathscr{K}_n$ of (2.16).

Define the discrete collocation method for (5.5) by

$$v_n = P_n \mathscr{G}(y + \mathscr{L}_n v_n) \tag{5.10a}$$

$$z_n = y + \mathscr{L}_n v_n \tag{5.10b}$$

For the actual implementation of (5.10a), we take

$$v_n(t) = \sum_{j=1}^{d} \gamma_j \phi_j(t) \qquad t \in D \tag{5.11a}$$

with $\{\gamma_j\}$ determined by solving

$$\sum_{j=1}^{d} \gamma_j \phi_j(t_i) = g\Big(t_i, y(t_i) + \sum_{j=1}^{d} \gamma_j \sum_{k=1}^{R} w_k L(t_i, \tau_k) \phi_j(\tau_k)\Big) \qquad i = 1, \ldots, d_n \tag{5.11b}$$

The error analysis for (5.10a) is the same as for the earlier discrete collocation method of Section 3. We use the assumptions following (5.6); and then we proceed as in Theorem 4. The errors in v_n and z_n satisfy

$$u_* - v_n = [I - P_n \mathscr{G}'(x_*)\mathscr{L}_n]^{-1}\{(I - P_n)u_* + P_n \mathscr{G}'(x_*)(\mathscr{L} - \mathscr{L}_n)u_* + \mathrm{O}(\|\mathscr{L}u_* - \mathscr{L}_n v_n\|^2) \tag{5.12}$$

$$x_* - z_n = \mathscr{L}u_* - \mathscr{L}_n v_n = (\mathscr{L} - \mathscr{L}_n)u_* + \mathscr{L}_n(u_* - v_n) \tag{5.13}$$

As part of the convergence proof of Theorem 4, it is shown that

$$\|[I - P_n \mathscr{G}'(x_*)\mathscr{L}_n]^{-1}\| \leq c < \infty, \qquad n \geq N \tag{5.14}$$

for some N. As a consequence, we have

$$\|u_* - v_n\|_\infty \leq c\, \|(I - P_n)u_* + P_n \mathscr{G}'(x_*)(\mathscr{L} - \mathscr{L}_n)u_*\|_\infty \qquad n \geq N \tag{5.15}$$

This involves the same type of errors as were encountered in Section 3 and Section 4, namely the interpolation error $(I - P_n)u_*$ and the numerical integration error $(\mathscr{L} - \mathscr{L}_n)u_*$.

To consider possible superconvergence of z_n to x_*, we note from (5.13) that the critical quantity to be considered is $\mathscr{L}_n(u_* - v_n)$. Using the identity

$$[I - P_n\mathscr{G}'(x_*)\mathscr{L}_n]^{-1} = I + [I - P_n\mathscr{G}'(x_*)\mathscr{L}_n]^{-1}P_n\mathscr{G}'(x_*)\mathscr{L}_n$$

and combining it with (5.12), we obtain

$$\mathscr{L}_n(u_* - v_n) = \{I + \mathscr{L}_n[I - P_n\mathscr{G}'(x_*)\mathscr{L}_n]^{-1}P_n\mathscr{G}'(x_*)\}\mathscr{L}_n(I - P_n)u_* + O(\|(\mathscr{L} - \mathscr{L}_n)u_*\|) + O(\|(I - P_n)u_*\|^2)$$

Combined with earlier results, including (5.13), we have

$$\|x_* - z_n\|_\infty = O(\|\mathscr{L}_n(I - P_n)u_*\|_\infty) + O(\|(\mathscr{L} - \mathscr{L}_n)u_*\|_\infty) + O(\|(I - P_n)u_*\|_\infty^2) \tag{5.16}$$

The error $\|\mathscr{L}_n(I - P_n)u_*\|_\infty$ can be studied for superconvergence, just as was done in Section 4 for the numerical integral operator $\mathscr{K}_n$ [e.g. see (4.15) and following]. For the types of interpolation and numerical integration methods analyzed in the first half of Section 4, those results are the same for our situation with $\|\mathscr{L}_n(I - P_n)u_*\|_\infty$ and $x_* - z_n$ and we do not repeat them here.

In the special case that $d_n = R_n$ and that the interpolation nodes $\{t_i\}$ are the same as the integration nodes $\{\tau_i\}$, we have that the iterated method

$$\hat{v}_n \equiv \mathscr{G}(y + \mathscr{L}_n v_n)$$

is the solution of the Nyström method

$$\hat{v}_n \equiv \mathscr{G}(y + \mathscr{L}_n \hat{v}_n)$$

Consequently, from the theory in Atkinson (1973) or Weiss (1974),

$$\|u_* - \hat{v}_n\|_\infty \leqslant c\|(\mathscr{L} - \mathscr{L}_n)u_*\|_\infty \qquad n \geqslant N$$

Since v_n and $\hat{v}_n$ coincide at the nodes, we have

$$z_n(t) = y(t) + \sum_{i=1}^{R} w_i L(t, t_i)v_n(t_i) = y(t) + \mathscr{L}_n\hat{v}_n(t_i)$$

From this and some simple manipulations, we have

$$\|x_* - z_n\|_\infty = O(\|(\mathscr{L} - \mathscr{L}_n)u_*\|_\infty)$$

Thus the case with $d_n = R_n$ is much simpler than the general case with the error (5.16).

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Added in proof: For work related to section 5, on the Hammerstein equation, see H. Brunner, "On implicitly linear and iterated collocation methods for Hammerstein integral equations", *Journal of Integral Equations and Applications* **3** (1991).