EIGENVECTOR CALCULATION

Let A have an approximate eigenvalue λ, so that $A - \lambda I$ is almost singular. How do we find a corresponding eigenvector? If the eigenvalue is of multiplicity 1, then in linear algebra courses we usually just try to solve the linear system

$$(A - \lambda I)\, x = 0$$

Oversimplifying, we usually drop one of the equations, arbitrarily assign one of the components of x, and then solve for the remaining components.

In numerical computations, this almost always leads to at least one of the eigenvectors being obtain inaccurately due to solving an ill-conditioned linear system. Consequently, another approach is usually used to find the eigenvector x.

INVERSE ITERATION

Let λ be an approximate eigenvalue of A, corresponding to some true eigenvalue λ_k for A. Let x_k denote the associated eigenvector. Choose an initial estimate $z^{(0)} \approx x_k$, often using a random number generator. The *inverse iteration* method is defined by

$$\begin{aligned} (A - \lambda I)\, w^{(m+1)} &= z^{(m)} \\ z^{(m+1)} &= \frac{w^{(m+1)}}{\left\| w^{(m+1)} \right\|_\infty} \end{aligned}$$

for $m = 0, 1, 2, ...$ It is important that λ not be exactly a true eigenvalue, or else the matrix $A - \lambda I$ will be singular.

EXAMPLE

Let

$$A = \begin{bmatrix} 6 & 4 & 3 \\ 4 & 3 & 2 \\ 3 & 2 & 1 \end{bmatrix}$$

The eigenvalues are

$$\begin{aligned} \lambda_1 &\doteq 0.23357813629678 \\ \lambda_2 &\doteq -0.42027581011042 \\ \lambda_3 &\doteq 10.18669767381363 \end{aligned}$$

The corresponding normalized eigenvectors are

$$\begin{bmatrix} -0.64061130860597 \\ 1.00000000000000 \\ -0.10198831463966 \end{bmatrix}, \quad \begin{bmatrix} 0.37938985766816 \\ 0.14105311855297 \\ -1.00000000000000 \end{bmatrix}$$

$$\begin{bmatrix} 1.00000000000000 \\ 0.68921959524450 \\ 0.47660643094521 \end{bmatrix}$$

In all cases, with a random $z^{(0)}$, we had $z^{(1)}$ was the correct answer to the digits shown.

CONVERGENCE

Let A have a diagonal Jordan canonical form,

$$P^{-1}AP = D = \text{diag}\,[\lambda_1, ..., \lambda_n]$$

with

$$P = [x_1, ..., x_n]$$

the matrix of eigenvectors (assumed to be normalized with $\|x_i\|_\infty = 1$).

For the given $z^{(0)}$, expand it in the basis of eigenvectors $\{x_1, ..., x_n\}$:

$$z^{(0)} = \sum_{j=1}^{n} \alpha_j x_j$$

Let $\lambda \approx \lambda_k$ for some k; and for simplicity, assume λ_k is a simple eigenvalue of A. Moreover, assume $\alpha_k \neq 0$. This is generally assured by using a random number generator to define $z^{(0)}$.

What are the eigenvalues and eigenvectors of $A - \lambda I$?

$$(A - \lambda I)\, x_i = (\lambda_i - \lambda)\, x_i, \quad i = 1, ..., n$$

Also, this implies

$$(A - \lambda I)^{-1} x_i = \frac{1}{\lambda_i - \lambda} x_i, \quad i = 1, ..., n$$

Our *inverse iteration* method

$$\begin{aligned} (A - \lambda I)\, w^{(m+1)} &= z^{(m)} \\ z^{(m+1)} &= \frac{w^{(m+1)}}{\left\| w^{(m+1)} \right\|_\infty} \end{aligned}$$

is a power method. Simply write it in the form

$$w^{(m+1)} = (A - \lambda I)^{-1} z^{(m)}$$

In §9.2 on the power method, replace the role of A by $(A - \lambda I)^{-1}$.

Then from (9.2.4) of that section,

$$z^{(m)} = \sigma_m \frac{(A - \lambda I)^{-m} z^{(0)}}{\left\| (A - \lambda I)^{-m} z^{(0)} \right\|_\infty}, \quad m \geq 0$$

with $|\sigma_m| = 1$. Using this formula and the earlier

$$(A - \lambda I)^{-1} x_i = \frac{1}{\lambda_i - \lambda} x_i, \quad i = 1, ..., n$$

we have

$$\begin{aligned}
(A - \lambda I)^{-m} z^{(0)} &= \sum_{j=1}^{n} \left(\frac{1}{\lambda_j - \lambda} \right)^m \alpha_j x_j \\
&= \left(\frac{1}{\lambda_k - \lambda} \right)^m \left\{ \alpha_k x_k + \sum_{\substack{j=1 \\ j \neq k}}^{n} \left(\frac{\lambda_k - \lambda}{\lambda_j - \lambda} \right)^m \alpha_j x_j \right\}
\end{aligned}$$

Then

$$z^{(m)} = \sigma_m \frac{\alpha_k x_k + \sum\limits_{\substack{j=1 \\ j \neq k}}^{n} \left(\frac{\lambda_k - \lambda}{\lambda_j - \lambda} \right)^m \alpha_j x_j}{\left\| \alpha_k x_k + \sum\limits_{\substack{j=1 \\ j \neq k}}^{n} \left(\frac{\lambda_k - \lambda}{\lambda_j - \lambda} \right)^m \alpha_j x_j \right\|_\infty}$$

If

$$|\lambda_k - \lambda| \ll \left| \lambda_j - \lambda \right|, \quad j \neq k$$

then

$$z^{(m)} \approx \sigma_m x_k$$

with only a small value of m. The error with iteration decreases by a factor of

$$\max_{j \neq k} \left| \frac{\lambda_k - \lambda}{\lambda_j - \lambda} \right|$$

This explains the rapid convergence of our numerical example.