EIGENVALUES OF TRIDIAGONAL MATRICES

$$A = \begin{bmatrix} \alpha_1 & \beta_1 & 0 & 0 & \cdots & 0 \\ \beta_1 & \alpha_2 & \beta_2 & 0 & & \\ 0 & \beta_2 & \alpha_3 & \beta_3 & & \vdots \\ & & \ddots & \ddots & \ddots & \\ \vdots & & & \beta_{n-2} & \alpha_{n-1} & \beta_{n-1} \\ 0 & & \cdots & & \beta_{n-1} & \alpha_n \end{bmatrix}$$

To calcuate the characteristic polynomial, we define the sequence of polynomials

$$f_k(\lambda) = \det \begin{bmatrix} \alpha_1 - \lambda & \beta_1 & 0 & \cdots & 0 \\ \beta_1 & \alpha_2 - \lambda & \beta_2 & & \vdots \\ 0 & & \ddots & & \\ \vdots & & & \alpha_{k-1} - \lambda & \beta_{k-1} \\ 0 & \cdots & & \beta_{k-1} & \alpha_k - \lambda \end{bmatrix}$$

for $k \geq 1$, with $f_1(\lambda) = \alpha_1 - \lambda$. Also introduce $f_0(\lambda) \equiv 1$. We will assume all $\beta_i \neq 0$.

We can obtain a recursive relationship for these polynomials. Expand the determinant by minors, using the last row. With that, there is a need for one further expansion in the last column of one of the reduced determinants. Then

$$f_k(\lambda) = (\alpha_k - \lambda)\, f_{k-1}(\lambda) - \beta_{k-1}^2 f_{k-2}(\lambda), \quad k \geq 1 \tag{1}$$

The characteristic polynomial for the original matrix T is $f_n(\lambda)$, and we want to compute its zeros.

Note that we can use (1) to evaluate $f_n(\lambda)$. What is the cost?
Assume the quantities $\left\{\beta_k^2\right\}$ have been prepared already. Then given a value of λ, $f_1(\lambda)$ costs 1 operation; and

$$f_2(\lambda) = (\alpha_k - \lambda)\, f_1(\lambda) - \beta_1^2$$

costs 3 operations. All of the remaining polynomials $f_k(\lambda)$ cost 4 operations, $k = 3, ..., n$. Thus there is a total operations cost of $4\,(n-1)$. This is more efficient than if we were to obtain $f_n(\lambda)$ explicitly.

We can solve

$$f_n(\lambda) = 0$$

by using a rootfinding method. Since it is difficult to obtain the derivative, the *secant method* is the natural choice for the root finding. Where are the roots located? We can use the Gerschgorin circle theorem to obtain a bounding interval. But there turns out to be a better approach.

The sequence $\{f_0, f_1, ..., f_n\}$ forms a *Sturm sequence* of polynomials; and such sequences have special properties. Given a point b, calculate

$$\{f_0(b), f_1(b), ..., f_n(b)\} \tag{2}$$

and observe the signs of these quantities. If some $f_j(\lambda) = 0$, then choose the sign of $f_j(\lambda)$ to be opposite to that of $f_{j-1}(\lambda)$. It can be shown that

$$f_j(\lambda) = 0 \quad \Rightarrow \quad f_{j-1}(\lambda) \neq 0$$

Having obtain a sequence of signs from (2), let $s(\lambda)$ denote the number of agreements of sign between consecutive members of the sign sequence.

EXAMPLE

From the text on p. 620, let

$$T = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

Then

$$f_0(\lambda) \equiv 1, \quad f_1(\lambda) = 2 - \lambda$$

$$f_k(\lambda) = (2 - \lambda)\, f_{k-1}(\lambda) - f_{k-2}(\lambda), \quad k = 2, ..., 6$$

Then for $\lambda = 1$,

$$\{f_0(1), f_1(1), ..., f_6(1)\} = \{1, 1, 0, -1, -1, 0, 1\}$$

The sign sequence is

$$\{+, +, -, -, -, +, +\}$$

Then $s(1) = 4$.

Theorem: The number of roots greater than $\lambda = a$ is given by $s(a)$. For $a < b$, the number of roots in the interval $a < \lambda \le b$ is given by $s(a) - s(b)$.

It can also be shown that with our assumption that all $\beta_i \neq 0$ that the matrix T will have n distinct eigenvalues

$$\lambda_1 < \lambda_2 < \cdots < \lambda_n$$

However, these may be located very close to one another. We can use the above theorem to separate the roots of $f_n(\lambda) = 0$ into disjoint subintervals; and then we can use a guaranteed rootfinder such as Brent's *zero* program to converge quickly to the root in each such subinterval. This is a practical method to find the roots; although it is most widely used when only a few eigenvalues are desired, say for example, the 5 largest ones.

EXAMPLE (continuation)

Recall the earlier example for

$$T = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\{f_0(1), f_1(1), ..., f_6(1)\} = \{1, 1, 0, -1, -1, 0, 1\}$$

with $s(1) = 4$. For $\lambda = 3$,

$$\{f_0(3), f_1(3), ..., f_6(3)\} = \{1, -1, 0, 1, -1, 0, 1\}$$

and $s(3) = 2$. Note that neither $\lambda = 1$ nor $\lambda = 3$ is a root; and $s(1) - s(3) = 2$. Therefore there are 2 roots in the interval $1 < \lambda < 3$.

This example is carried further in the text on pages 622-623.