THE POWER METHOD

Assume the matrix A has a diagonal Jordan canonical form,

$$P^{-1}AP = D = \text{diag}[\lambda_1, ..., \lambda_n]$$

Moreover, assume there is a single dominant eigenvalue, say

$$|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_n| \geq 0$$

Then we want to find λ_1 and its corresponding eigenvector x_1.

From the similarity relation, the columns of P, which we will call $x_1, ..., x_n$, are a basis of our vector space $\mathbb{C}^n$. Of course, we also have

$$Ax_i = \lambda_i x_i, \quad i = 1, ..., n$$

Let $z^{(0)}$ be an arbitrarily chosen initial guess for x_1. Then we can write

$$z^{(0)} = \sum_{j=1}^{n} \alpha_j x_j$$

We assume that $\alpha_1 \neq 0$. The fundamental idea is quite simple. Apply A repeatedly to the vector $z^{(0)}$. Then

$$A^k z^{(0)} = \sum_{j=1}^{n} \alpha_j A^k x_j = \sum_{j=1}^{n} \alpha_j \lambda_j^k x_j$$

Factor out λ_1^k,

$$A^k z^{(0)} = \lambda_1^k \left[\alpha_1 x_1 + \sum_{j=2}^{n} \alpha_j \left(\frac{\lambda_j}{\lambda_1} \right)^k x_j \right]$$

Since λ_1 is dominant, the fractions

$$\left(\frac{\lambda_j}{\lambda_1} \right)^k \rightarrow 0 \quad \text{as} \quad k \rightarrow \infty$$

Thus

$$A^k z^{(0)} \approx \alpha_1 \lambda_1^k x_1$$

for larger values of k. Also, as k increases to $k+1$, the value of $A^k z^{(0)}$ increases by approximately a factor of λ_1. These ideas are the basis of the power method. Implementing it is a bit more difficult.

ALGORITHM

Given $z^{(0)}$, define

$$w^{(1)} = Az^{(0)}$$

Let β_1 be the component of $w^{(1)}$ of maximum size,

$$|\beta_1| = \left\| w^{(1)} \right\|_\infty$$

and define

$$z^{(1)} = \frac{w^{(1)}}{\beta_1}$$

Continue on in the same manner, defining next $w^{(2)}$, β_2, and $z^{(2)}$, and so on.

$$w^{(2)} = Az^{(1)}$$

$$|\beta_2| = \left\| w^{(2)} \right\|_\infty$$

$$z^{(2)} = \frac{w^{(2)}}{\beta_2}$$

Choose a component, say ℓ, which we will use for computing an eigenvalue approximation. Then we define

$$\lambda_1^{(m)} = \frac{w_\ell^{(m)}}{z_\ell^{(m-1)}}, \quad m = 1, 2, 3, ...$$

In the book, we show that under suitable assumptions,

$$\left\| z^{(m)} - \sigma_m \frac{x_1}{\|x_1\|_\infty} \right\|_\infty \le c \left| \frac{\lambda_2}{\lambda_1} \right|^m, \quad m \ge 0$$

for some σ_m, with $|\sigma_m| = 1$. Also,

$$\lambda_1^{(m)} = \lambda_1 \left[1 + O\left(\left| \frac{\lambda_2}{\lambda_1} \right|^m \right) \right], \quad m \ge 0$$

ACCELERATION METHODS

There are several methods to accelerate the convergence of the iterates $\lambda_1^{(k)}$ and $z^{(m)}$, including Aitken extrapolation.

For the case that A is symmetric, one of the better known methods is the Rayleigh quotient. In this case, define

$$\lambda_1^{(m)} = \frac{\left(Az^{(m)}, z^{(m)}\right)}{\left(z^{(m)}, z^{(m)}\right)}, \quad m \geq 0$$

Then it can be shown that

$$\lambda_1^{(m)} = \lambda_1 \left[1 + O\left(\left|\frac{\lambda_2}{\lambda_1}\right|^{2m}\right)\right], \quad m \geq 0$$

so that the errors decrease by a factor of

$$\left(\frac{\lambda_2}{\lambda_1}\right)^2$$

with each iteration.