# GENERAL THEORY

Return to questions of convergence and stability of the multistep method

$$y_{n+1} = \sum_{j=0}^{p} a_j y_{n-j} + h \sum_{j=-1}^{p} b_j f(x_{n-j}, y_{n-j}), \quad n \geq p$$

Earlier, this was shown to be stable and convergent if it was consistent and if all $a_j \geq 0$. We now examine necessary and sufficient conditions.

# CONSISTENCY

*Consistency* means that

$$\frac{1}{h} \max_{x_p \leq x_n \leq b} |T_n(Y)| \to 0 \quad \text{as} \quad h \to 0$$

## STABILITY

To examine the *stability*, we assume a new set of initial values, say $z_0, ..., z_p$ with

$$|y_i - z_i| \leq \epsilon, \quad i = 0, 1, ..., p$$

Note this is to be true for all sufficiently small values of $h$, say $h \leq h_0$ for some $h_0 > 0$. Then the method is stable for our particular differential equation $y' = f(x, y)$ if we can prove the resulting numerical solutions $\{y_i\}$ and $\{z_i\}$ satisfy

$$\max_{x_0 \leq x_n \leq b} |y_i - z_i| \leq c\epsilon$$

for all $h \leq h_0$ and for some constant $c$ which depends on $f$ but not on $h$. We say the method is stable if this is true for all well-defined differential equation problems.

# CONVERGENCE

To examine the *convergence* in solving

$$y' = f(x, y), \quad x_0 \le x_n \le b, \quad y(x_0) = Y_0$$

for $Y(x)$, we first introduce the error in the initial values $y_0, ..., y_p$:

$$\eta(h) = \max_{i=0,1,\ldots,p} |Y(x_0 + ih) - y_h(x_0 + ih)|$$

We say the method is convergent for a particular differential equation if $\eta(h) \to 0$ implies

$$\max_{x_0 \le x_n \le b} |Y(x_i) - y_h(x_i)| \to 0$$

We say the method is convergent if the above is true for well-behaved differential equations.

# THE ROOT CONDITION

Introduce the polynomial

$$\rho(r) = r^{p+1} - \sum_{j=0}^{p} a_j r^{p-j}$$

Note that the consistency condition

$$\sum_{j=0}^{p} a_j = 1$$

implies $\rho(1) = 0$. We denote $r_0 = 1$; and we denote the remaining roots of $\rho(r)$ by $r_1, ..., r_p$.

*The Root Condition*: The method satisfies

1. $\left|r_j\right| \leq 1$ for all $j$; and
2. If $\left|r_j\right| = 1$, then $\rho'(r_j) \neq 0$.

# THE MODEL EQUATION

Consider the initial value problem

$$y' = f(x, y), \quad x_0 \le x \le b, \quad y(x_0) = Y_0$$

When $x$ is close to $x_0$, expand $f(x, Y(x))$ in a Taylor series about $x_0$:

$$\begin{aligned} f(x, Y(x)) &= f(x_0, Y_0) + f_x(x_0, Y_0)\,(x - x_0) \\ &\quad + f_y(x_0, Y_0)\,(Y(x) - Y_0) + \cdots \end{aligned}$$

Thus for the differential equation,

$$\begin{aligned} Y'(x) &= f(x, Y(x)) \\ &= f(x_0, Y_0) + f_x(x_0, Y_0)\,(x - x_0) \\ &\quad + f_y(x_0, Y_0)\,(Y(x) - Y_0) + \cdots \end{aligned} \tag{1}$$

For $x \approx x_0$, we drop the higher order terms, corresponding to powers

$$(x - x_0)^i\,(Y(x) - Y_0)^j$$

with $i + j \ge 2$.

Introduce the new unknown function

$$V(x) = Y(x) - Y_0$$

and look at the equation based on the linear Taylor polynomial terms in (1). Thus we have

$$V'(x) \doteq f(x_0, Y_0) + f_x(x_0, Y_0)\,(x - x_0) + f_y(x_0, Y_0)V(x)$$

Introduce $\lambda = f_y(x_0, Y_0)$ and

$$g(x) = f(x_0, Y_0) + f_x(x_0, Y_0)\,(x - x_0)$$

Then $V$ satisfies

$$V'(x) \doteq \lambda V(x) + g(x) \qquad (2)$$

This is called the *model equation* for the original problem $y' = f(x, y)$.

When studying stability, we are interested in perturbations of the equation or the initial value, and we study the resulting change in the solution by looking at differences in the equations. For example, suppose

$$V'(x) = \lambda V(x) + g(x), \quad V(x_0) = V_0$$

$$V_\epsilon'(x) = \lambda V_\epsilon(x) + g(x), \quad V_\epsilon(x_0) = V_0 + \epsilon$$

Then we study the size of $W = V_\epsilon - V$, and we do so by subtracting:

$$W' = \lambda W, \quad W(x_0) = \epsilon \tag{3}$$

This is the model equation for studying stability; and (2) is the model equation for questions of convergence.

# SYSTEMS - MODEL EQUATION

For the system

$$\mathbf{y}' = \mathbf{f}(x, \mathbf{y}), \quad x_0 \le x \le b, \quad \mathbf{y}(x_0) = \mathbf{Y}_0$$

of some order $m > 1$, we can can carry out an analysis similar to the above. It leads to the model equation

$$\mathbf{V}'(x) \doteq \Lambda \mathbf{V}(x) + \mathbf{g}(x) \tag{4}$$

with $\Lambda = \mathbf{f}_\mathbf{y}(x_0, \mathbf{Y}_0)$ and

$$\mathbf{g}(x) = \mathbf{f}(x_0, \mathbf{Y}_0) + \mathbf{f}_x(x_0, \mathbf{Y}_0)\,(x - x_0)$$

Recall that $\mathbf{f}_\mathbf{y}(x, \mathbf{y})$ is the Jacobian matrix of $\mathbf{f}_\mathbf{y}(x, \mathbf{y})$:

$$(\mathbf{f}_\mathbf{y}(x, \mathbf{y}))_{i,j} = \frac{\partial f_i(x, y_1, ..., y_m)}{\partial y_j}, \quad 1 \le i, j \le m$$

With most differential equations, the model equation (4) can be converted to the set of model equations

$$z_i' = \lambda_i z_i + \gamma_i(x), \quad i = 1, ..., m$$

with $\lambda_1, ..., \lambda_m$ the eigenvalues of $\mathbf{f}_\mathbf{y}(x_0, \mathbf{Y}_0)$.

# SOLVING THE MODEL EQUATION

We solve the model equation

$$y' = \lambda y, \quad y(0) = 1 \tag{5}$$

where I have changed back to the standard notation with the letter $y$. The multistep method

$$y_{n+1} = \sum_{j=0}^{p} a_j y_{n-j} + h \sum_{j=-1}^{p} b_j f(x_{n-j}, y_{n-j}), \quad n \geq p$$

becomes

$$y_{n+1} = \sum_{j=0}^{p} a_j y_{n-j} + h \sum_{j=-1}^{p} b_j \lambda y_{n-j}, \quad n \geq p \tag{6}$$

This is a *linear difference equation* of order $p + 1$. Review the material for the midpoint method in which we solved the second order linear difference equation

$$y_{n+1} = y_{n-1} + 2h\lambda y_n, \quad n \geq 1$$

Assume a solution of the form

$$y_n = r^n, \quad n \geq 0$$

Then $r$ must satisfy

$$r^{n+1} = \sum_{j=0}^{p} a_j r^{n-j} + h\lambda \sum_{j=-1}^{p} b_j r^{n-j}, \quad n \geq p$$

Since we can assume $r \neq 0$, we multiply by $r^{p-n}$, obtaining

$$r^{p+1} = \sum_{j=0}^{p} a_j r^{p-j} + h\lambda \sum_{j=-1}^{p} b_j r^{p-j}, \quad n \geq p \quad (7)$$

Introduce

$$\sigma(r) = \sum_{j=-1}^{p} b_j r^{p-j}$$

and recall

$$\rho(r) = r^{p+1} - \sum_{j=0}^{p} a_j r^{p-j} \quad (8)$$

Then

$$r^{p+1} = \sum_{j=0}^{p} a_j r^{p-j} + h\lambda \sum_{j=-1}^{p} b_j r^{p-j}, \quad n \geq p$$

can be written as

$$\rho(r) - h\lambda\sigma(r) = 0 \tag{9}$$

This is called the *characteristic equation* for our multistep method.

Recall we denote the roots of $\rho(r) = 0$ by

$$r_0 = 1, r_1, \cdots, r_p$$

We will denote the roots of (9) by

$$r_0(h\lambda), \cdots, r_p(h\lambda) \tag{10}$$

with $r_0(h\lambda)$ the root satisfying $r_0(0) = 1$, and all $r_j(0) = r_j$.

The general solution of our linear difference equation

$$r^{n+1} = \sum_{j=0}^{p} a_j r^{n-j} + h\lambda \sum_{j=-1}^{p} b_j r^{n-j}, \quad n \geq p$$

is given by

$$y_n = \beta_0 \left[r_0(h\lambda)\right]^n + \sum_{j=1}^{p} \beta_j \left[r_j(h\lambda)\right]^n, \quad n \geq 0 \tag{11}$$

The coefficients $\beta_0, \cdots, \beta_p$ are chosen so the formula (11) agrees with the given initial values $y_0, \cdots, y_p$.

We use this to study the convergence and stability of the multistep method. It will turn out that

$$\beta_0 \to 1 \quad \text{as} \quad h \to 0$$

$$\beta_1, \cdots, \beta_p \to 0 \quad \text{as} \quad h \to 0$$

The term

$$\sum_{j=1}^{p} \beta_j \left[r_j(h\lambda)\right]^n$$

is a *parasitic solution*.

# MAIN RESULTS

*Theorem*: If the multistep method is consistent, then it is convergent if and only if the characteristic roots $r_0,\ldots,r_p$ satisfy the root condition.

*Theorem*: If the multistep method is consistent, then it is stable if and only if the characteristic roots $r_0,\ldots,r_p$ satisfy the root condition.

*Corollary:* If the multistep method is consistent, then it is stable if and only if it is convergent.

*The Root Condition*: The method satisfies
1. $\left|r_j\right| \leq 1$ for all $j$; and
2. If $\left|r_j\right| = 1$, then $\rho'(r_j) \neq 0$.

$$\rho(r) = r^{p+1} - \sum_{j=0}^{p} a_j r^{p-j}$$

# EXAMPLE OF INSTABILITY

The method

$$y_{n+1} = 10y_n - 9y_{n-1} + h\left[-3y'_n - 5y'_{n-1}\right], \quad n \geq 1$$

has order 2. For it,

$$\begin{aligned} \rho(r) &= r^2 - a_0 r - a_1 \\ &= r^2 - 10r + 9 \end{aligned}$$

The roots are $r_0 = 1$, $r_1 = 9$. Look at solving the very simple problem

$$y' = 0, \quad y(0) = 0$$

Clearly the exact solution is $Y(x) \equiv 0$.

The numerical method is

$$y_{n+1} = 10y_n - 9y_{n-1}, \quad n \geq 1$$

If we choose the initial conditions

$$y_0 = y_1 = 0$$

then we get the numerical solution $y_n = 0$, all $n \geq 0$. Now choose

$$z_0 = \frac{\epsilon}{9}, \quad z_1 = \epsilon$$

for some small $\epsilon > 0$. Thus

$$\max \{|y_0 - z_0|, |y_1 - z_1|\} = \epsilon$$

But the solution of

$$z_{n+1} = 10z_n - 9z_{n-1}, \quad n \geq 1$$

in this case is

$$z_n = \epsilon\, 9^{n-1}, \quad n \geq 0$$

At the point $x = 1$, we have $n = 1/h$. Thus

$$\max_{0 \leq x_n \leq 1} |y_n - z_n| = \frac{\epsilon}{9}\, 9^{1/h}$$

This "blows up" as $h \to 0$.

*Reviewing*, when our multistep method

$$y_{n+1} = \sum_{j=0}^{p} a_j y_{n-j} + h \sum_{j=-1}^{p} b_j f(x_{n-j}, y_{n-j}), \quad n \ge p$$

is applied to the model problem

$$y' = \lambda y, \quad y(0) = 1$$

we have the general solution

$$y_n = \beta_0 \left[r_0(h\lambda)\right]^n + \sum_{j=1}^{p} \beta_j \left[r_j(h\lambda)\right]^n, \quad n \ge 0$$

The choice of coefficients $\left\{\beta_j\right\}$ depends on the initial conditions $y_0,\cdots,y_p$. Also, $r_0(h\lambda),\cdots,r_p(h\lambda)$ are the roots of the characteristic equation

$$\rho(r) - h\lambda\sigma(r) = 0$$

How the solution behaves depends on the values of these roots.

# CONVERGENCE AND STABILITY

In the text, we prove the stability theorem for the model equation

$$y' = \lambda y + g(x), \quad x \geq 0, \quad y(0) = Y_0$$

We prove the convergence theorem for only the special case

$$y' = \lambda y, \quad x \geq 0, \quad y(0) = Y_0$$

General proofs are beyond the scope of this course, as they require the theory of non-homogeneous linear difference equations, along with methods for handling the general case of

$$y' = f(x, y), \quad y(x_0) = Y_0$$

# WEAK STABILITY

The term

$$\sum_{j=1}^{p} \beta_j \left[r_j(h\lambda)\right]^n$$

is the parasitic portion of the solution. We want it to be well-behaved in comparison to the *dominant* term

$$\beta_0 \left[r_0(h\lambda)\right]^n$$

To accomplish this, we ask that

$$\left|r_j(h\lambda)\right| \leq \left|r_0(h\lambda)\right|, \quad j = 1, ..., p$$

If this is true for all small values of $|h\lambda|$, we say the multistep method is *strongly stable*.

This is accomplished if the characteristic roots satisfy the *The Strong Root Condition*:

$$\left|r_j\right| < 1, \quad j = 1, ..., p$$

# EXAMPLES

For the *midpoint method*,

$$\rho(r) = r^2 - 1$$

and $r_0 = 1$, $r_1 = -1$. It does not satisfy the strong root condition. We also know that

$$|r_1(h\lambda)| > |r_0(h\lambda)|$$

when $\lambda < 0$, making the midpoint method only weakly stable.

For the Adams-Bashforth method of order 2,

$$y_{n+1} = y_n + \frac{h}{2}\left[3f(x_n, y_n) - f(x_{n-1}, y_{n-1})\right], \quad n \geq 1$$

the characteristic polynomial is

$$\rho(r) = r^2 - r = r(r-1)$$

Thus we have $r_0 = 1$, $r_1 = 0$. It satisfies the strong root condition, and easily

$$|r_1(h\lambda)| \leq |r_0(h\lambda)|$$

for all small values of $|h\lambda|$. For the general Adams-Bashforth and Adams-Moulton methods of order $p+1$, we have

$$\rho(r) = r^{p+1} - r^p = r^p(r-1)$$

We easily have the strong root condition satisfied.

## STABILITY REGIONS

We often say something is true for a method if the stepsize $h$ is chosen sufficiently small. In the multiple inequality

$$\left|r_j(h\lambda)\right| \leq |r_0(h\lambda)|\,, \quad j = 1, ..., p$$

we ask what are the values of $h\lambda$ for which it is true. This is called the *region of relative stabilty* for the multistep method. Methods of the same order of convergence can than be compared as to which has the larger region of relative stability.

The region of relative stability is not as well studied as another stability region, one called the *region of absolute stability*. To introduce it, we return to Euler's method and investigate further its stability.

# EXAMPLE: A RELATIVE STABILITY REGION

For the Adams-Bashforth method of order 2,

$$y_{n+1} = y_n + \frac{h}{2}\left[3f(x_n, y_n) - f(x_{n-1}, y_{n-1})\right], \quad n \geq 1$$

the characteristic equation is

$$r^2 = r + \frac{h\lambda}{2}[3r - 1]$$

With reference to past definitions,

$$\rho(r) = r^2 - r, \quad \sigma(r) = \frac{1}{2}[3r - 1]$$

The characteristic roots are

$$r_0(h\lambda) = \frac{1}{2}\left[1 + \frac{3}{2}h\lambda + \left(1 + h\lambda + \frac{9}{4}(h\lambda)^2\right)^{\frac{1}{2}}\right]$$

$$r_1(h\lambda) = \frac{1}{2}\left[1 + \frac{3}{2}h\lambda - \left(1 + h\lambda + \frac{9}{4}(h\lambda)^2\right)^{\frac{1}{2}}\right]$$

We want

$$|r_1(h\lambda)| \leq |r_0(h\lambda)|$$

For real $\lambda$, this is accomplished with $-\frac{2}{3} \leq h\lambda$.

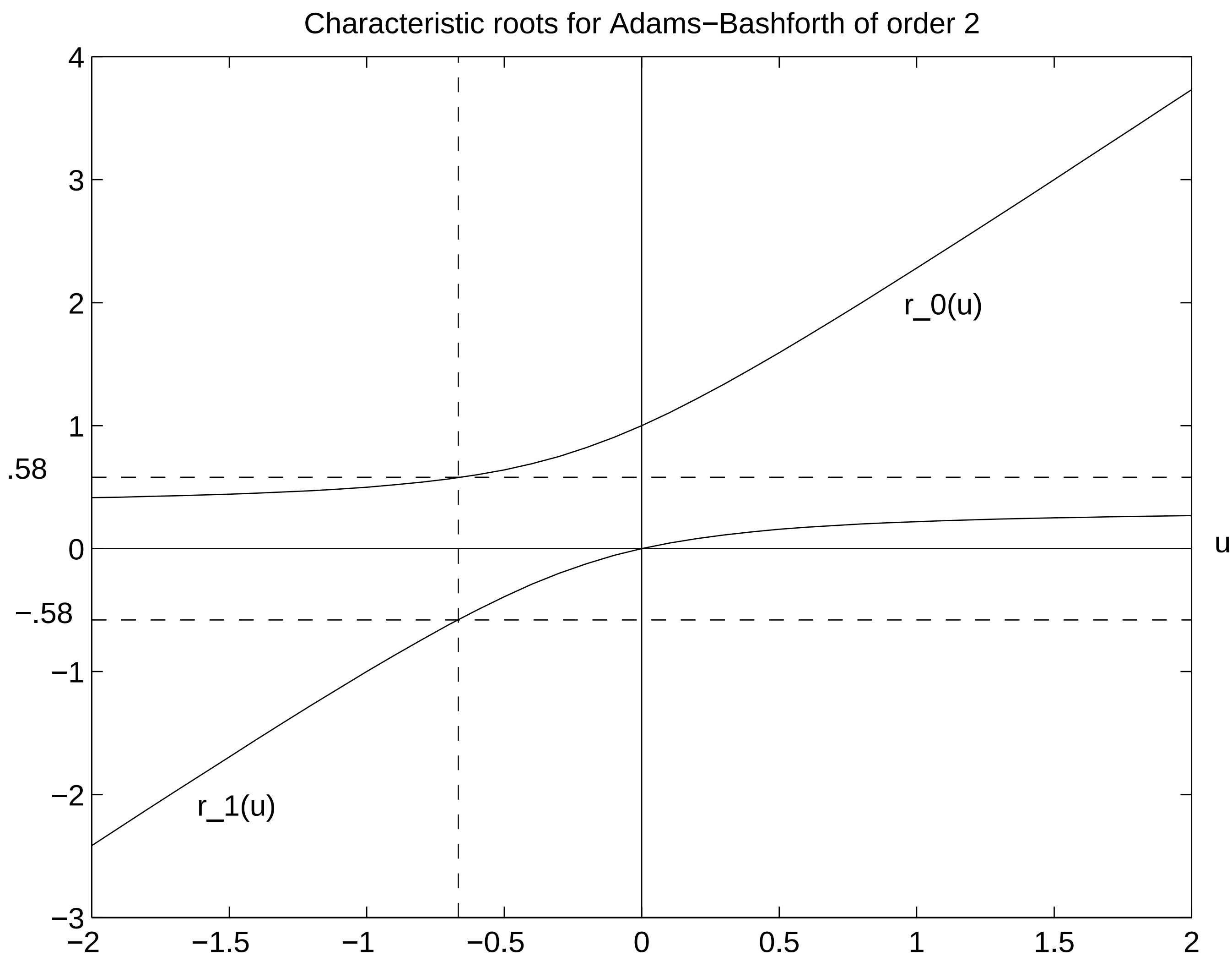


Figure 1: $r_0(u) \geq |r_1(u)|$ for $u \geq -\frac{2}{3}$

# EULER'S METHOD

Solve the model equation

$$y' = \lambda y + g(x), \quad x \geq 0, \quad y(0) = Y_0$$

using Euler's method

$$y_{n+1} = y_n + h\left[\lambda y_n + g(x_n)\right], \quad n \geq 0, \quad y_0 = Y_0$$

Now consider the perturbed problem

$$z_{n+1} = z_n + h\left[\lambda z_n + g(x_n)\right], \quad n \geq 0, \quad z_0 = Y_0 + \epsilon$$

For the original differential equation problem, this leads to a perturbed solution $Z(x)$ with

$$Z(x) - Y(x) = \epsilon\, e^{\lambda x}$$

We want to consider the cases $\lambda < 0$ or real$(\lambda) < 0$. Then

$$Z(x) - Y(x) \to 0 \quad \text{as} \quad x \to \infty$$

What happens to the numerical solution?

Let $e_n = z_n - y_n$; and $e_0 = \epsilon$. The numerical solution $\{e_n\}$ satisfies

$$e_{n+1} = e_n + h\lambda e_n = (1 + h\lambda)\, e_n$$

This implies

$$e_n = (1 + h\lambda)^n \,\epsilon, \quad n \geq 0$$

In imitation of the behaviour of perturbed solution in the original problem, we would like to have

$$e_n \rightarrow 0 \quad \text{as} \quad x_n \rightarrow \infty$$

or equivalently as $n \rightarrow \infty$. This is true if and only if

$$|1 + h\lambda| < 1$$

$$|h\lambda - (-1)| < 1$$

Thus it is true for all complex numbers $h\lambda$ which are within a distance of 1 of the point -1. We call the set

$$\{u \mid |u + 1| < 1\}$$

the *region of absolute stability* for Euler's method.

# THE BACKWARD EULER METHOD

Recall the backward Euler method

$$y_{n+1} = y_n + hf(x_{n+1}, y_{n+1}), \quad n \geq 0$$

This is an implicit one-step method. It truncation method is

$$T_n(Y) = -\frac{h^2}{2} Y''(\xi_n)$$

for some $x_n \leq \xi_n \leq x_{n+1}$. As with earlier multistep methods (cf. §6.3), we can show convergence and standard stability. In particular,

$$\begin{aligned} |Y(x_n) - y_h(x_n)| \;\; &\leq \;\; e^{2K(x_n - x_0)} |Y(x_0) - y_h(x_0)| \\ &\quad + h \frac{e^{2K(x_n - x_0)} - 1}{4K} \|Y''\|_\infty \end{aligned}$$

for all values of $h$ satisfying $2hK \leq 1$, with $K$ the usual Lipschitz constant for $f(x, y)$.

Apply the backward Euler method to the model problem

$$y' = \lambda y + g(x), \quad x \geq 0, \quad y(0) = Y_0$$

obtaining

$$y_{n+1} = y_n + h\left[\lambda y_{n+1} + g(x_{n+1})\right], \quad n \geq 0, \quad y_0 = Y_0$$

Now consider the perturbed problem

$$z_{n+1} = z_n + h\left[\lambda z_{n+1} + g(x_{n+1})\right], \quad n \geq 0, \quad z_0 = Y_0 + \epsilon$$

As before, we subtract and then examine

$$e_{n+1} = e_n + h\lambda e_{n+1}, \quad e_0 = \epsilon$$

$$(1 - h\lambda)e_{n+1} = e_n$$

$$e_{n+1} = \left(\frac{1}{1 - h\lambda}\right) e_n$$

with $e_0 = \epsilon$. By straightforward induction,

$$e_n = \left(\frac{1}{1 - h\lambda}\right)^n \epsilon$$

Assuming $\lambda$ is real and negative, we have $1-h\lambda > 1$. Therefore, $e_n \to 0$ as $x_n \to \infty$. For $\lambda$ complex, write $\lambda = \alpha + i\beta$, and assume $\alpha < 0$. Then

$$|1-h\lambda| = |1-h\alpha - ih\beta| = \left[(1-h\alpha)^2 + (h\beta)^2\right]^{\frac{1}{2}}$$

This shows

$$|1-h\lambda| > 1$$

and again $e_n \to 0$ as $x_n \to \infty$.

We say the region of absolute stability for the backward Euler method consists of the entire left half of the complex plane. All numerical methods with such a stability region are called *A-stable*. This will end up making the backward Euler method especially desirable for *stiff* problems, those for which real$(\lambda) < 0$ and is very large in magnitude, say real$(\lambda) = -10,000$.

# CONNECTIONS TO CONVERGENCE

Return to the model equation

$$y' = \lambda y + g(x), \quad x \geq 0, \quad y(0) = Y_0$$

and its solution by Euler's method

$$y_{n+1} = y_n + h\left[\lambda y_n + g(x_n)\right], \quad n \geq 0, \quad y_0 = Y_0$$

For the true solution,

$$\begin{aligned} Y(x_{n+1}) &= Y(x_n) + hY'(x_n) + \frac{h^2}{2} Y''(\xi_n) \\ &= Y(x_n) + h\left[\lambda Y(x_n) + g(x_n)\right] + \frac{h^2}{2} Y''(\xi_n) \end{aligned}$$

Subtract Euler's method, obtaining

$$e_{n+1} = (1 + h\lambda)\, e_n + \frac{h^2}{2} Y''(\xi_n)$$

with $e_0 = Y_0 - y_0 = 0$.

To simplify making our point, assume the true solution is $Y(x) = x^2$ (meaning $g(x)$ has been chosen appropriately). Then the error equation is

$$e_{n+1} = (1 + h\lambda)\, e_n + h^2, \quad e_0 = 0$$

This yields

$$\begin{aligned}
e_1 &= h^2 \\
e_2 &= (1 + h\lambda)\, h^2 + h^2 \\
e_3 &= \Big[(1 + h\lambda)^2 + (1 + h\lambda) + 1\Big]\, h^2 \\
e_n &= \Big[(1 + h\lambda)^{n-1} + (1 + h\lambda)^{n-2} + \cdots \\
&\qquad + (1 + h\lambda) + 1]\, h^2
\end{aligned}$$

$$\begin{aligned}
e_n &= \frac{(1 + h\lambda)^n - 1}{(1 + h\lambda) - 1} h^2 \\
&= \frac{(1 + h\lambda)^n - 1}{\lambda} h
\end{aligned}$$

Note what happens as $n \to \infty$ in the case $|1 + h\lambda| > 1$ versus what happens when $|1 + h\lambda| < 1$. In the latter case, we have $e_n \to 0$; but in the former, we have $e_n$ will “blow up”.

# GENERAL REMARKS

It is necessary to have $h\lambda$ belong to the absolute stability region in order to have sensible behaviour in the convergence of the numerical method. *But*, note that the truncation error in this case is simply

$$T_n(Y) = h^2$$

which does not depend on $\lambda$. Thus the size of $\lambda$ affects the convergence, even when it is not involved in the truncation error for the problem. The stepsize $h$ may need to be very small, even though that is not needed in order to have the truncation error be small. With the backward Euler method, there is not such requirement. For the case of a general derivative $f(x, y)$, recall that the role of $\lambda$ is played by $f_y(x, y)$.

One way in which multistep methods are compared is by comparing their regions of absolute stability. For methods of equivalent accuracy, the method with the larger stability region is considered superior. By this criteria, Adams-Moulton methods are considered superior to Adams-Bashforth methods.

## REGION OF ABSOLUTELY STABILITY

How do we define the *region of absolute stability* and compute it? We are looking at the model equation

$$y' = \lambda y, \quad y(0) = 1$$

Apply the numerical method to get

$$y_{n+1} = \sum_{j=0}^{p} a_j y_{n-j} + h\lambda \sum_{j=-1}^{p} b_j y_{n-j}, \quad n \geq p \tag{12}$$

We want to have the numerical solution satisfy

$$y_n \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty \tag{13}$$

When is this guaranteed to happen?

The general solution to (12) is given by

$$y_n = \sum_{j=0}^{p} \beta_j \left[ r_j(h\lambda) \right]^n, \quad n \geq 0$$

The result (13) follows if we have

$$\left| r_j(h\lambda) \right| < 1, \quad j = 0, 1, ..., p \tag{14}$$

The set of all numbers $h\lambda$ which satisfy simultaneously all of these inequalities form the region of absolute stability of the numerical method.

# EXAMPLE

For the Adams-Bashforth method of order 2,

$$y_{n+1} = y_n + \frac{h}{2}\left[3f(x_n, y_n) - f(x_{n-1}, y_{n-1})\right], \quad n \geq 1$$

the characteristic roots are

$$r_0(h\lambda) = \frac{1}{2}\left[1 + \frac{3}{2}h\lambda + \left(1 + h\lambda + \frac{9}{4}(h\lambda)^2\right)^{\frac{1}{2}}\right]$$

$$r_1(h\lambda) = \frac{1}{2}\left[1 + \frac{3}{2}h\lambda - \left(1 + h\lambda + \frac{9}{4}(h\lambda)^2\right)^{\frac{1}{2}}\right]$$

We want

$$|r_0(h\lambda)| < 1, \quad |r_1(h\lambda)| < 1$$

This region is shown in the graph as the $k = 2$ case (cf, p. 407)

# THE TRAPEZOIDAL METHOD

Recall the trapezoidal method

$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_{n+1})\right], \quad n \geq 0$$

This is a one-step method; and it is left as an assigned problem to show that it is an A-stable method. For a numerical example, see p. 409.

It would seem a good idea to look for A-stable methods of order higher than the trapezoidal method. Unfortunately, a result of G. Dahlquist shows that there are no multistep methods of order greater than two that are A-stable.