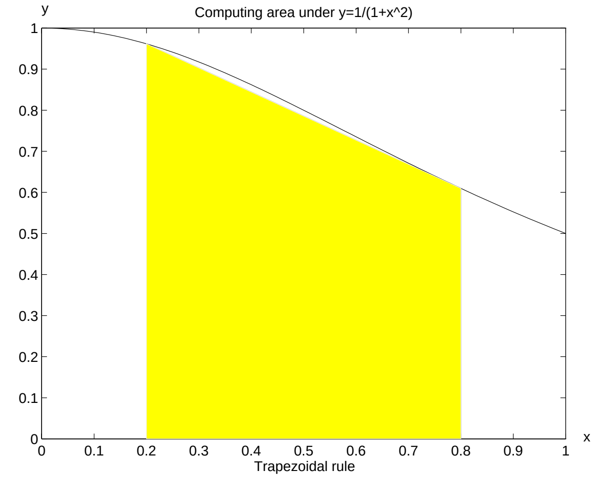

THE TRAPEZOIDAL QUADRATURE RULE

From Chapter 5, we have the quadrature formula

$$\int_a^b g(x)\,dx \;=\; \left(\frac{b-a}{2}\right)[g(a)+g(b)] - \frac{(b-a)^3}{12} g''(\xi)$$

for some $a \le \xi \le b$.

THE TRAPEZOIDAL RULE FOR ODEs

Integrate

$$Y'(x) = f(x, Y(x))$$

over the interval $[x_n, x_{n+1}]$:

$$Y(x_{n+1}) = Y(x_n) + \int_{x_n}^{x_{n+1}} f(x, Y(x))\, dx$$

Apply the trapezoidal quadrature rule to this integral:

$$\begin{aligned} Y(x_{n+1}) &= Y(x_n) \\ &\quad + \frac{h}{2}\left[f(x_n, Y(x_n)) + f(x_{n+1}, Y(x_{n+1})\right] \\ &\quad - \frac{h^3}{12} Y'''(\xi_n) \end{aligned}$$

Dropping the error term, we have the *trapezoidal rule*:

$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_{n+1})\right]$$

$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_{n+1})\right], \quad n \geq 0$$

This is a one-step implicit method. Its truncation error is

$$T_n(Y) = -\frac{h^3}{12} Y'''(\xi_n)$$

For its error, we can use the general theory of §6.3; or directly,

$$\begin{aligned} |Y(x_n) - y_h(x_n)| \quad \leq \quad & e^{2K(x_n - x_0)} \, |Y_0 - y_h(x_0)| \\ & + \frac{e^{2K(x_n - x_0)} - 1}{K} \cdot \frac{h^2}{12} \, \|Y'''\|_\infty \end{aligned}$$

provided $hK \leq 1$ and the Lipschitz condition is satisfied.

STABILITY

Let

$$|y_0 - z_0| \le \epsilon$$

and

$$z_{n+1} = z_n + \frac{h}{2}\left[f(x_n, z_n) + f(x_{n+1}, z_{n+1})\right], \quad n \ge 0$$

Then

$$|y_n - z_n| \le \epsilon \, e^{2K(x_n - x_0)}, \quad x_0 \le x_n \le b$$

ASYMPTOTIC ERROR FORMULA

We can show

$$Y(x_n) - y_h(x_n) = D(x_n)h^2 + O(h^3)$$

$$D'(x) = f_y(x, Y(x))D(x) - \frac{1}{12}Y'''(x), \quad D(x_0) = 0$$

assuming $y_0 = Y_0$.

RICHARDSON EXTRAPOLATION

At any node point x common to the use of stepsizes h and $2h$, we have

$$Y(x) - y_h(x) = D(x)h^2 + O(h^3)$$

$$Y(x) - y_{2h}(x) = 4D(x)h^2 + O(h^3)$$

Then

$$Y(x) - y_{2h}(x) \doteq 4\,[Y(x) - y_h(x)]$$

$$Y(x) \doteq y_h(x) + \frac{1}{3}\,[y_h(x) - y_{2h}(x)]$$

ITERATIVE SOLUTION

To solve for y_{n+1} in

$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_{n+1})\right]$$

define

$$y_{n+1}^{(j+1)} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_{n+1}^{(j)})\right]$$

for $j = 0, 1, 2, \ldots$, with an initial guess $y_{n+1}^{(0)} \approx y_{n+1}$. To analyze the convergence:

$$\begin{aligned}
y_{n+1} - y_{n+1}^{(j+1)} &= \frac{h}{2}\left[f(x_{n+1}, y_{n+1}) - f(x_{n+1}, y_{n+1}^{(j)})\right] \\
\left|y_{n+1} - y_{n+1}^{(j+1)}\right| &= \frac{h}{2}\left|f(x_{n+1}, y_{n+1}) - f(x_{n+1}, y_{n+1}^{(j)})\right| \\
&\leq \frac{hK}{2}\left|y_{n+1} - y_{n+1}^{(j)}\right|, \quad j \geq 0
\end{aligned}$$

According to this, convergence is assured if we choose h so small that

$$\frac{hK}{2} < 1$$

If we use the mean value theorem, we get a better idea of nature of the convergence:

$$y_{n+1} - y_{n+1}^{(j+1)} \doteq \frac{h}{2}\frac{\partial f(x_{n+1}, y_{n+1})}{\partial y}\left[y_{n+1} - y_{n+1}^{(j)}\right]$$

for $j = 0, 1, 2, ...$ Thus the crucial factor is

$$\frac{h}{2}\frac{\partial f(x_{n+1}, y_{n+1})}{\partial y}$$

and we need its magnitude to be less than 1. The smaller this factor, the faster the convergence. Also, note that with most stable differential equations, the partial derivative is negative. Thus the error in the iterates will oscillate between negative and positive, which means the iterates are oscillating about the desired solution y_{n+1}.

CHOOSING THE INITIAL GUESS

We generally to choose

$$y_{n+1} - y_{n+1}^{(0)} = O(h^p)$$

for some $p > 0$. How to choose $y_{n+1}^{(0)}$?

(a) The preceding answer:

$$y_{n+1}^{(0)} = y_n$$

(b) Euler's method:

$$y_{n+1}^{(0)} = y_n + hf(x_n, y_n)$$

(c) Midpoint method:

$$y_{n+1}^{(0)} = y_{n-1} + 2hf(x_n, y_n)$$

THE LOCAL SOLUTION

An important idea to consider is that of the *local solution* of the differential equation. Given that we have gotten to the point (x_n, y_n) with the trapezoidal method, consider the initial value problem

$$y' = f(x, y), \quad x \geq x_n, \quad y(x_n) = y_n$$

Denote the solution of this problem by $u_n(x)$. It is the exact solution of the differential equation from x_n onwards, based on our best knowledge of the solution at x_n. Thus

$$u_n'(x) = f(x, u_n(x)), \quad x \geq x_n, \quad u_n(x_n) = y_n$$

It is this solution that we are truly estimating at each step.

If we proceed in analogy with the derivation of Euler's method,

$$\begin{aligned} u_n(x_{n+1}) &= u_n(x_n) + h u_n'(x_n) + \frac{h^2}{2} u_n''(\xi_n) \\ &= y_n + h f(x_n, y_n) + \frac{h^2}{2} u_n''(\xi_n) \end{aligned}$$

If we let

$$y_{n+1}^{(0)} = y_n + h f(x_n, y_n)$$

then

$$u_n(x_{n+1}) - y_{n+1}^{(0)} = \frac{h^2}{2} u_n''(\xi_n)$$

Similarly, for the derivation of the trapezoidal method,

$$\begin{aligned} u_n(x_{n+1}) &= u_n(x_n) \\ &\quad + \frac{h}{2}\left[f(x_n, u_n(x_n)) + f(x_{n+1}, u_n(x_{n+1}))\right] \\ &\quad\quad - \frac{h^3}{12} u_n'''(\zeta_n) \\ &= y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, u_n(x_{n+1}))\right] \\ &\quad\quad - \frac{h^3}{12} u_n'''(\zeta_n) \end{aligned}$$

From this, we can derive

$$u_n(x_{n+1}) - y_{n+1} = -\frac{h^3}{12} u_n'''(x_n) + O(h^4)$$

as is described in the text.

If we use the Euler predictor,

$$y_{n+1}^{(0)} = y_n + hf(x_n, y_n)$$

then

$$\begin{aligned} y_{n+1} - y_{n+1}^{(0)} &= \frac{h^2}{2}u_n''(\xi_n) + \frac{h^3}{12}u_n'''(x_n) + O(h^4) \\ &= \frac{h^2}{2}u_n''(\xi_n) + O(h^3) \end{aligned}$$

Return to error in the iteration,

$$\left| y_{n+1} - y_{n+1}^{(j+1)} \right| \le \frac{hK}{2} \left| y_{n+1} - y_{n+1}^{(j)} \right|, \quad j \ge 0$$

Then

$$\left| y_{n+1} - y_{n+1}^{(1)} \right| \le O(h^3)$$

$$\left| y_{n+1} - y_{n+1}^{(2)} \right| \le O(h^4)$$

Usually we try to make the iteration error less significant than the truncation error in the trapezoidal method,

$$T_n(Y) = -\frac{h^3}{12} Y'''(\xi_n)$$

Thus we would use two iterations of the trapezoidal iteration equation

$$y_{n+1}^{(j+1)} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_{n+1}^{(j)})\right], \quad j = 0, 1$$

and then let $y_{n+1} = y_{n+1}^{(2)}$.

If we repeat this discussion with the midpoint predictor,

$$y_{n+1}^{(0)} = y_{n-1} + 2hf(x_n, y_n)$$

then we can derive

$$u_n(x_{n+1}) - y_{n+1}^{(0)} = \frac{5h^3}{12} u_n'''(x_n) + O(h^4)$$

Combined with the local error for the trapezoidal rule

$$u_n(x_{n+1}) - y_{n+1} = -\frac{h^3}{12} u_n'''(x_n) + O(h^4)$$

we have

$$y_{n+1} - y_{n+1}^{(0)} = \frac{h^3}{2} u_n'''(x_n) + O(h^4)$$

For the iteration, we need iterate only once, obtaining

$$\left| y_{n+1} - y_{n+1}^{(1)} \right| \leq O(h^4)$$

We then proceed with $y_{n+1} = y_{n+1}^{(1)}$.

A-STABILITY

For reasons examined in a later section (§6.9), we look at the behaviour of numerical solutions to

$$y' = \lambda y, \quad y(0) = 1$$

when λ is a real or complex number with real$(\lambda) < 0$. The true solution is $Y(x) = e^{\lambda x}$; and as $x \to \infty$, $Y(x) \to 0$. We ask when the numerical solution has the same behaviour.

The trapezoidal method for this case is

$$y_{n+1} = y_n + \frac{h}{2}\left[\lambda y_n + \lambda y_{n+1}\right]$$

We can solve for y_{n+1} to get

$$y_{n+1} = \left(\frac{1 + \dfrac{h\lambda}{2}}{1 - \dfrac{h\lambda}{2}} \right) y_n, \qquad y_0 = 1$$

Together with $y_0 = 1$, this leads to

$$y_n = \left(\frac{1 + \frac{h\lambda}{2}}{1 - \frac{h\lambda}{2}} \right)^n, \quad n \geq 0$$

If $\lambda < 0$ and is real, then clearly $y_n \to 0$ as $n \to \infty$. By a slightly more complicated argument, the same is true if real$(\lambda) < 0$.

Numerical methods for which this is true, independent of the size of h, are called *A-stable methods*. A-stable methods are very useful is solving *stiff* differential equations, which we explore in §6.9.