# A DIFFERENTIATION RESULT

Using arguments based on the use of Taylor series, we can show

$$g'(a) = \frac{g(a+h) - g(a-h)}{2h} - \frac{h^2}{6} g'''(\xi)$$

for some $a - h \leq \xi \leq a + h$. To persuade yourself of this, write

$$\begin{aligned} g(a+h) &= g(a) + hg'(a) + \frac{h^2}{2} g''(a) + \frac{h^3}{6} g'''(\xi_+) \\ g(a-h) &= g(a) - hg'(a) + \frac{h^2}{2} g''(a) - \frac{h^3}{6} g'''(\xi_-) \end{aligned}$$

with $a - h \leq \xi_- \leq a \leq \xi_+ \leq a + h$. Subtract and solve for $g'(a)$, obtaining

$$\begin{aligned} g'(a) &= \frac{g(a+h) - g(a-h)}{2h} \\ &\qquad - \frac{h^2}{12} \left[ g'''(\xi_+) + g'''(\xi_-) \right] \end{aligned}$$

We can use the intermediate value theorem to complete the proof othe result.

# THE MIDPOINT METHOD

Apply the above result with $g(x) = Y(x)$, and use $Y'(x) = f(x, Y(x))$. Solve for $Y(x+h)$ to obtain

$$Y(x+h) = Y(x-h) + 2hf(x, Y(x)) + \frac{h^3}{3} Y'''(\xi)$$

with some $x - h \le \xi \le x + h$. This leads directly to the midpoint numerical method:

$$y_{n+1} = y_{n-1} + 2hf\left(x_n, y_n\right), \quad n = 1, 2, 3, ...$$

We also have directly that the truncation error is given by

$$T_n(Y) = \frac{h^3}{3} Y'''(\xi_n)$$

for some $x_{n-1} \le \xi_n \le x_{n+1}$.

If we analyze this directly, we can obtain the error bound

$$\max_{x_0 \le x_j \le b} \left| Y(x_j) - y_h(x_j) \right| \le c_1 \eta(h; Y) + c_2 \frac{h^2}{3} \left\| Y''' \right\|_\infty$$

$$\eta(h; Y) = \max \left\{ \left| Y(x_0) - y_h(x_0) \right|, \left| Y(x_1) - y_h(x_1) \right| \right\}$$

$$c_1 = e^{2K(b - x_0)}, \quad c_2 = \frac{c_1 - 1}{2K}$$

## STABILITY OF MIDPOINT METHOD

We can also prove the following stability result. Let

$$|y_0 - z_0|, |y_1 - z_1| \le \epsilon$$

Then consider the two numerical solutions

$$y_{n+1} = y_{n-1} + 2hf(x_n, y_n), \quad n = 1, 2, 3, ...$$

$$z_{n+1} = z_{n-1} + 2hf(x_n, z_n), \quad n = 1, 2, 3, ...$$

Then we can show

$$|y_n - z_n| \le \epsilon\, e^{2K(x_n - x_0)}$$

Unfortunately, there are still aspects of the midpoint method which make it unsuitable, even when compared with Euler's method.

# AN ILLUSTRATIVE EXAMPLE

Consider solving the initial value problem

$$y' = \lambda y, \quad x \geq 0, \quad y(0) = 1$$

This has the true solution $Y(x) = e^{\lambda x}$. The midpoint method becomes

$$y_{n+1} = y_{n-1} + 2h\lambda y_n, \quad n = 1, 2, 3, \ldots$$

We want the exact solution of this *linear difference equation* or *finite difference equation*.

# LINEAR DIFFERENCE EQUATIONS

A general *linear difference equation* has the form

$$y_{n+1} = a_0 y_n + a_1 y_{n-1} + \cdots + a_p y_{n-p} + g_n, \quad n \geq p \tag{1}$$

This is called a linear difference equation of *order* $p+1$. If $g_n \equiv 0$, then we say it is a *homogeneous* equation; and otherwise it is inhomogeneous. The theory for these equations runs parallel to that for linear differential equations of the form

$$y^{(p+1)} = a_0 y^{(p)} + \cdots + a_p y + g(x)$$

For the homogeneous case ($g(x) \equiv 0$), we look for solutions of the form $e^{\lambda x}$; and then we combine them to get the general solution of the differential equation. Note that we can write

$$e^{\lambda x} = r^x, \quad r = e^{\lambda}$$

With (1), we look for a solution of the form

$$y_n = r^n, \quad n \geq 0$$

Then we combine them to get a general solution.

# EXAMPLE - CONTINUED

Recall the equation

$$y_{n+1} = y_{n-1} + 2h\lambda y_n, \quad n = 1, 2, 3, \ldots \tag{2}$$

Assume

$$y_n = r^n, \quad n \geq 0$$

Then

$$r^{n+1} = r^{n-1} + 2h\lambda r^n, \quad n \geq 1$$

Clearly $r = 0$ leads to a solution, but it is only the zero solution. Otherwise, we can cancel $r^{n-1}$ to get

$$r^2 = 1 + 2h\lambda r$$

This has two solutions:

$$r_0 = h\lambda + \left(1 + h^2\lambda^2\right)^{\frac{1}{2}}, \quad r_1 = h\lambda - \left(1 + h^2\lambda^2\right)^{\frac{1}{2}}$$

Then we assert the general solution of (2) is given by

$$y_n = \beta_0 r_0^n + \beta_1 r_1^n, \quad n \geq 0$$

In solving (2), we have given initial values $y_0 = 1$ and $y_1 \doteq e^{\lambda h}$. We use these to determine the coefficients $\beta_0$ and $\beta_1$.

$$\beta_0 + \beta_1 = y_0$$

$$\beta_0 r_0 + \beta_1 r_1 = y_1$$

This yields

$$\beta_0 = \frac{y_1 - r_1 y_0}{r_0 - r_1}, \quad \beta_1 = \frac{r_0 y_0 - y_1}{r_0 - r_1}$$

If we choose

$$y_0 = 1, \quad y_1 = e^{\lambda h}$$

then

$$\beta_0 = 1 + O(h^2 \lambda^2), \quad \beta_1 = O(h^3 \lambda^3)$$

We also need to understand something more about $r_0^n$ and $r_1^n$.
Using

$$(1+u)^{\frac{1}{2}} = 1 + \frac{1}{2}u - \frac{1}{8}u^2 + O(u^3)$$

we can obtain

$$r_0^n = e^{\lambda x_n}\left[1 + O(h)\right]$$

$$r_1^n = (-1)^n e^{-\lambda x_n}\left[1 + O(h)\right]$$

Thus

$$\begin{aligned} y_n &= \beta_0 r_0^n + \beta_1 r_1^n \\ &= \left[1 + O(h^2\lambda^2)\right] e^{\lambda x_n}\left[1 + O(h)\right] \\ &\quad + O(h^3\lambda^3)\left\{(-1)^n e^{-\lambda x_n}\left[1 + O(h)\right]\right\} \end{aligned}$$

The term $r_0^n$ corresponds to the true solution $e^{\lambda x_n}$ that we are seeking. The term $r_1^n$ is what we term a *parasitic solution*, and it does not correspond to anything occurring in the original differential equation.

When do we have trouble from the parasitic solution in

$$y_n = \beta_0 r_0^n + \beta_1 r_1^n$$

The answer is when $\lambda < 0$. In this case,

$$r_0 = h\lambda + \left(1 + h^2\lambda^2\right)^{\frac{1}{2}}, \quad r_1 = h\lambda - \left(1 + h^2\lambda^2\right)^{\frac{1}{2}}$$

$$0 < r_0 < 1, \quad r_1 < -1$$

and then $r_1^n$ dominates $r_0^n$. The parasitic solution grows in size, and eventually it dominates the solution we are seeking, which arises from the term $r_0^n$. In addition to becoming larger in size, the term $r_1^n$ will also oscillate in sign, and thus eventually the solution $y_n$ will also begin to oscillate.

When methods have this form of instability, even though they are stable in the original sense, then we say they are only *weakly stable*.

# GENERALIZATION

For a general equation $y' = f(x,y)$, how do we decide if a similar behaviour will occur when using the midpoint method? The equation $y' = \lambda y$ is in fact a model equation for the general differential equation when we are studying questions of stability. If we return to Lecture 2, we find that the general equation can be approximated by

$$y' = \lambda y + g(x)$$

with

$$\lambda = \frac{\partial f(x_0, Y_0)}{\partial y}, \quad g(x) = f(x, Y(x))$$

If

$$\frac{\partial f(x, Y(x))}{\partial y} < 0$$

then we can expect difficulty from the midpoint method. These are exactly the cases which are most stable when solving the differential equation.