DIFFERENTIAL EQUATIONS

A principal model of physical phenomena.

The ordinary differential equation:

$$y' = f(x, y)$$

The initial value:

$$y(x_0) = Y_0$$

Find a solution $Y(x)$ on some interval $x_0 \leq x \leq b$. Together these two conditions constitute an "initial value problem".

We will study methods for solving systems of first order equations, but we begin with a single equation.

Many of the crucial ideas in the numerical analysis arise from properties of the original equation.

SPECIAL CASES

1. $y'(x) = \lambda y(x) + b(x),\ x \geq x_0$
   General solution:
   $$Y(x) = ce^{\lambda x} + \int_{x_0}^{x} e^{\lambda(x-t)} b(t)dt$$
   with c arbitrary. With $y(x_0) = Y_0$,
   $$Y(x) = Y_0 e^{\lambda(x-x_0)} + \int_{x_0}^{x} e^{\lambda(x-t)} b(t)dt$$

2. $y'(x) = ay^2$
   General solution:
   $$Y(x) = \frac{-1}{ax+c}, \quad c \text{ arbitrary}$$
   With $y(x_0) = Y_0 \neq 0$, use
   $$c = -ax_0 - \frac{1}{Y_0}$$

3. $y'(x) = -\left[y(x)\right]^2 + y(x)$
   General solution:

$$Y(x) = \frac{1}{1 + ce^{-x}}$$

4. Separable equations: $y'(x) = g(y(x))h(x)$
   General solution: Write

$$\frac{1}{g(y)}\frac{dy}{dx} \;=\; h(x)$$

   Let $z = y(x)$, $dz = y'(x)dx$. Evaluate the integrals in

$$\int \frac{dz}{g(z)} = \int h(x)dx$$

   Replace z by $Y(x)$ and solve for $Y(x)$, if possible.

DIRECTION FIELDS

At each point (x, y) at which the function f is defined, evaluate it to get $f(x, y)$. Then draw in a small line segement at this point with slope $f(x, y)$. With enough of these, we have a picture of how the solutions of the differential equation

$$y' = f(x, y)$$

behave.

Consider the differential equation

$$y' = -y + 2\cos x$$

We can draw direction fields by hand by the method described in the book; or we can use the Matlab program I am providing.

SOLVABILITY THEORY

Consider whether there is a function $Y(x)$ which satisfies

$$y' = f(x, y), \quad x \geq x_0, \quad y(x_0) = Y_0 \qquad (1)$$

Assume there is some "open" set D containing (x_0, Y_0) for which:

1. If two points (x, y) and (x, z) are contained in D, then the line segment joining them is also contained in D.

2. $f(x, y)$ is continuous for all points (x, y) contained in D.

3. $\partial f(x, y)/\partial y$ is continuous for all points (x, y) contained in D.

Then there is an interval $[c, d]$ containing x_0 and there is a unique function $Y(x)$ defined on $[c, d]$ which satisfies (1), with the graph of $Y(x)$ contained in D.

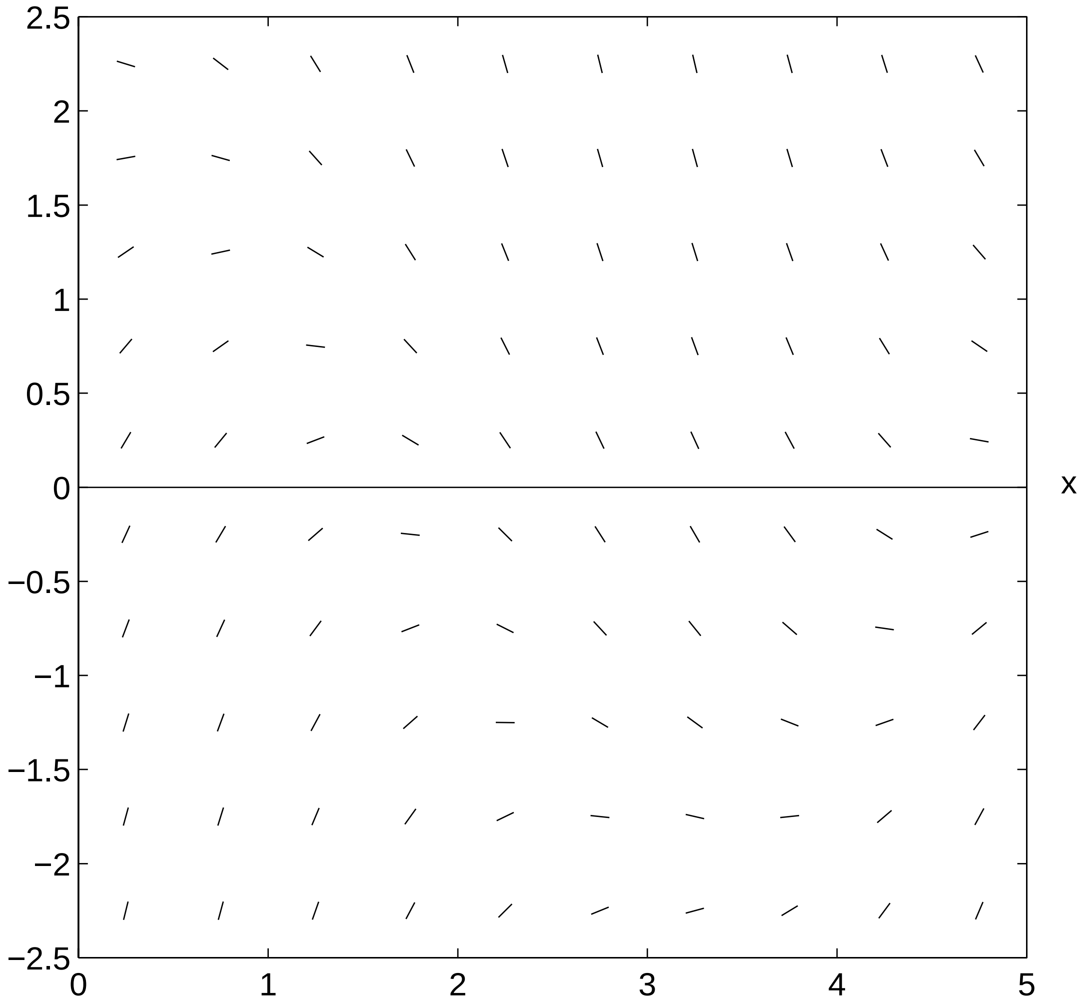

Figure 1: Direction field for $y' = -y + 2\cos x$

THE LIPSCHITZ CONDITION

The preceding condition on the partial derivative of f is an easy way to specify the following condition is satisfied, and it is the condition which is really needed.

<u>The Lipschitz condition</u>: There is a non-negative constant K for which

$$|f(x, y) - f(x, z)| \leq K\,|y - z|$$

for all points (x, y), (x, z) in the region D. In practice, we can use

$$K = \max_{(x,y)\in D} \left| \frac{\partial f(x, y)}{\partial y} \right|$$

The Lipschitz condition returns throughout our treatment of both the theory of differential equations and the theory of the numerical methods for their solution.

EXAMPLE

Let $\alpha > 0$ be a given constant, and consider solving

$$y' = \frac{2x}{\alpha^2} y^2, \quad x \geq 0, \quad y(0) = 1$$

Then the partial derivative is

$$f_y(x, y) = \frac{4xy}{\alpha^2}$$

and $f_y(0, 1) = 0$. Thus $f_y(x, y)$ is small for (x, y) near to $(0, 1)$, and it is continuous for all (x, y). Choose

$$D = \{(x, y) : |x| \leq 1, |y| \leq B\}$$

for some $B > 0$. Then there is a solution $Y(x)$ on some interval $[c, d]$ containing $x_0 = 0$. How big is $[c, d]$? In this case,

$$Y(x) = \frac{\alpha^2}{\alpha^2 - x^2}, \quad -\alpha < x < \alpha$$

If α is small, then the interval is small.

IMPROVED SOLVABILITY THEORY

Assume there is a Lipschitz constant K for which f satisfies

$$|f(x,y) - f(x,z)| \leq K\,|y - z|$$

for all (x,y), (x,z) satisfying

$$x_0 \leq x \leq b, \quad -\infty < y, z < \infty$$

Then the initial value problem

$$y' = f(x,y), \quad x_0 \leq x \leq b, \quad y(x_0) = Y_0$$

has a solution $Y(x)$ on the entire interval $x_0 \leq x \leq b$.

Example: Consider $y' = y + g(x)$ with $g(x)$ continuous for all x. Then

$$y' = y + g(x), \quad y(x_0) = Y_0$$

has a solution $Y(x)$ has a unique continuous solution for $-\infty < x < \infty$.

STABILITY

The concept of "stability" refers in a loose sense to what happens to the solution $Y(x)$ of an initial value problem if we make a small change in the "data", which includes both the differential equation and the initial value.

If small changes in the data lead to large changes in the solution, then we say the initial value problem is "unstable"; whereas if small changes in the data lead to small changes in the solution, we call the problem "stable".

EXAMPLE

Consider solving

$$y' = 100y - 101e^{-x}, \quad y(0) = 1 \tag{2}$$

This has a solution of $Y(x) = e^{-x}$.

Now consider the "perturbed problem"

$$y' = 100y - 101e^{-x}, \quad y(0) = 1 + \epsilon$$

where ϵ is some small number. The solution of this is $Y_\epsilon(x) = e^{-x} + \epsilon e^{100x}$, and

$$Y_\epsilon(x) - Y(x) = \epsilon e^{100x}$$

Thus $Y_\epsilon(x) - Y(x)$ increases very rapidly as x increases, and we say (2) is an "unstable" problem.

We must now define these concepts with a bit more care.

STABILITY

Consider the initial value problem

$$y' = f(x, y), \quad x_0 \leq x_0 \leq b, \quad y(x_0) = Y_0$$

and denote its solution by $Y(x)$. Now consider the problem

$$y' = f(x, y), \quad x_0 \leq x \leq b, \quad y(x_0) = Y_0 + \epsilon$$

and denote its solution by $Y(x; \epsilon)$. We want to study the behaviour of

$$Z(x; \epsilon) \equiv Y(x; \epsilon) - Y(x)$$

to see how it changes as x increases. To do this, we first derive an equation for $Z(x; \epsilon)$.

Substitute the solutions into the above equations:

$$Y'(x) = f(x, Y(x)), \quad Y(x_0) = Y_0 \tag{3}$$

$$Y'(x;\epsilon) = f(x, Y(x;\epsilon)), \quad Y(x_0;\epsilon) = Y_0 + \epsilon \tag{4}$$

both for $x_0 \le x \le b$. Subtracting (3) from (4), we have

$$Z'(x;\epsilon) = f(x, Y(x;\epsilon)) - f(x, Y(x)), \quad Z(x_0;\epsilon) = \epsilon$$

We can apply the mean-value theorem to the right side to obtain the "perturbation equation"

$$\begin{aligned} Z'(x;\epsilon) &= \frac{\partial f(x,\zeta)}{\partial y}\left[Y(x;\epsilon) - Y(x)\right], \quad Z(x_0;\epsilon) = \epsilon \\ &= \frac{\partial f(x,\zeta)}{\partial y} Z(x;\epsilon) \end{aligned} \tag{5}$$

where $\zeta \equiv \zeta(x;\epsilon)$ is some unknown number between $Y(x)$ and $Y(x;\epsilon)$.

In the case ϵ is small, and with x close to x_0, we have $\zeta \approx Y(x)$ and

$$Z'(x;\epsilon) \doteq \frac{\partial f(x, Y(x))}{\partial y} Z(x;\epsilon), \quad Z(x_0;\epsilon) = \epsilon$$

or even

$$Z'(x;\epsilon) \doteq \frac{\partial f(x, Y_0)}{\partial y} Z(x;\epsilon), \quad Z(x_0;\epsilon) = \epsilon$$

Note that both of these has the form

$$Z'(x) \doteq a(x)Z(x), \quad x \geq x_0; \quad Z(x_0) = \epsilon$$

a simple linear differential equation.

EXAMPLE

Consider solving

$$y' = a(x) + \cos y, \qquad y(0) = 0$$

In this

$$f(x, y) = a(x) + \cos y$$

$$\frac{\partial f(x, y)}{\partial y} = -\sin y$$

The equation (5) becomes

$$Z'(x;\epsilon) = -\sin(\zeta(x;\epsilon))\, Z(x;\epsilon), \quad x \geq 0; \quad Z(0,\epsilon) = \epsilon$$

with $\zeta(x;\epsilon)$ some unknown number between $Y(x)$ and $Y(x;\epsilon)$. This can be solved as

$$Z(x;\epsilon) = \epsilon \exp\left[-\int_0^x \sin\zeta(t;\epsilon)\, dt\right]$$

Since $-1 \leq \sin(\zeta(t;\epsilon)) \leq 1$, we have

$$|Z(x;\epsilon)| \leq |\epsilon|\, e^x, \quad x \geq 0$$

In general, the equation

$$Z'(x;\epsilon) = \frac{\partial f(x,\zeta)}{\partial y} Z(x;\epsilon), \quad x \geq x_0; \quad Z(x_0;\epsilon) = \epsilon$$

can be solved to get

$$Z(x;\epsilon) = \epsilon \exp\left[\int_{x_0}^{x} \frac{\partial f(t,\zeta(t;\epsilon))}{\partial y} dt\right], \quad x \geq x_0 \quad (6)$$

We can use this to derive a variety of results, depending on the assumptions we make about the partial derivative

$$f_y(x,y) \equiv \frac{\partial f(x,y)}{\partial y}$$

For example, suppose

$$\frac{\partial f(x,y)}{\partial y} \le 0, \quad x_0 \le x \le b, \quad -\infty < y < \infty$$

Then using the perturbation equation (6), we have

$$|Z(x;\epsilon)| \le |\epsilon|$$

because

$$\int_{x_0}^{x} \frac{\partial f(t, \zeta(t;\epsilon))}{\partial y} dt \le 0$$

$$e^{\text{negative quantity}} \le 1$$

Example: Consider the equation

$$y' = -y^3 + g(x)$$

Then

$$f_y(x,y) = -3y^2 \le 0$$

for all (x,y). For this equation, the perturbed solution satisfies

$$|Y(x;\epsilon) - Y(x)| \le |\epsilon|, \quad x \ge x_0$$

If all we know is that

$$K \equiv \max_{\substack{x_0 \le x \le b \\ -\infty < y < \infty}} \left| \frac{\partial f(x,y)}{\partial y} \right| < \infty$$

then we can say only that

$$\begin{aligned} |Z(x;\epsilon)| &\le |\epsilon| \exp\left[\int_{x_0}^{x} \left|\frac{\partial f(t,\zeta(t;\epsilon))}{\partial y}\right| dt\right] \\ &\le |\epsilon| \exp\left[\int_{x_0}^{x} K dt\right] \\ &= |\epsilon| \, e^{K(x-x_0)} \end{aligned} \tag{7}$$

This might increase quite rapidly if K is a large number, or even moderately large.

WHEN ARE WE IN TROUBLE?

If the partial derivative $f_y(x, y)$ evaluated at $y = Y(x)$, namely $f_y(x, Y(x))$ is large, then the perturbation

$$Z(x; \epsilon) = \epsilon \exp\left[\int_{x_0}^{x} \frac{\partial f(t, \zeta(t; \epsilon))}{\partial y} dt\right], \quad x \geq x_0 \quad (8)$$

will grow rapidly as x increases. In the problem

$$y' = \lambda y + g(x), \quad y(0) = Y_0 \quad (9)$$

we have $f_y(x, y) = \lambda$, and

$$Z(x; \epsilon) = \epsilon \text{ exp}\left[\int_{x_0}^{x} \lambda dt\right] = \epsilon\, e^{\lambda(x - x_0)}$$

If $\lambda < 0$, the perturbation becomes smaller with increasing x. But if $\lambda > 0$, then the perturbation increases as x increases; and if λ is large in magnitude, then the perturbation increases rapidly.

GENERAL DISCUSSION AND NOTATION

The words "stable" and "unstable" are not sufficiently descriptive of what happens in practice. In

$$y' = \lambda y + g(x), \quad y(0) = Y_0 \tag{10}$$

as λ increases, $\lambda > 0$, the perturbation becomes worse. In Chapter 1, we introduced the idea of *condition number*. Small condition numbers were associated with "well-conditioned problems", and large condition numbers were associated with "ill-conditioned problems". We could introduce condition numbers for our initial value problem

$$y' = f(x, y), \quad x_0 \le x \le b, \quad y(x_0) = Y_0$$

by using our perturbation equation for $Z(x; \epsilon)$. This is seldom done in practice. Instead we note that some problems are more stable or well-conditioned than others. With (10), we look at the size of λ. With a general problem, we look at the size of $f_y(x, y)$ for the values of y around $Y(x)$.

HIGHER ORDER EQUATIONS

Consider the third order equation

$$y'''(x) - 2y'y^2 = \cos x, \quad x \geq x_0$$

The initial conditions for this are specifications of $y(x_0)$, $y'(x_0)$, $y''(x_0)$.

To rewrite this as a system of first order equations, introduce

$$y_1 = y, \quad y_2 = y', \quad y_3 = y''$$

Then these new functions satisfy the first order system

$$\begin{array}{rcll} y_1' & = & y_2, & y_1(x_0) = y(x_0) \\ y_2' & = & y_3, & y_2(x_0) = y'(x_0) \\ y_3' & = & \cos x + 2y_2y_1^2, & y_3(x_0) = y''(x_0) \end{array}$$

We do other higher order equations in a similar manner.

EXAMPLE

Consider the vector differential equation

$$m\mathbf{r}''(t) = \frac{-mMG}{|\mathbf{r}(t)|^3}\mathbf{r}(t), \quad \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$$

Then we rewrite this as a system of first order equations by introducing

$$y_1 = x, \quad y_2 = x'$$

$$y_3 = y, \quad y_4 = y'$$

This leads to the first order system

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= \frac{-MG}{r^3}y_1 \\ y_3' &= y_4 \\ y_4' &= \frac{-MG}{r^3}y_3 \end{aligned}$$

with $r = \left(y_1^2 + y_3^2\right)^{\frac{1}{2}}$.

FIRST ORDER SYSTEMS

The general form of a system of m first order differential equations is

$$\begin{aligned} y_1' &= f_1(x, y_1, ..., y_m), \quad y_1(x_0) = Y_{1,0} \\ &\vdots \qquad\qquad\qquad\qquad\qquad \vdots \\ y_m' &= f_m(x, y_1, ..., y_m), \quad y_m(x_0) = Y_{m,0} \end{aligned}$$

We can make this look like our earlier single equation by introducing matrix-vector notation. Let

$$\mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}, \quad \mathbf{Y}_0 = \begin{bmatrix} Y_{1,0} \\ \vdots \\ Y_{m,0} \end{bmatrix}$$

$$\mathbf{f}(x, \mathbf{y}) = \begin{bmatrix} f_1(x, y_1, ..., y_m) \\ \vdots \\ f_m(x, y_1, ..., y_m) \end{bmatrix}$$

Then the linear system can be written as

$$\mathbf{y}' = \mathbf{f}(x, \mathbf{y}), \quad x \geq x_0; \quad \mathbf{y}(x_0) = \mathbf{Y}_0$$

The earlier role of $f_y(x,y)$ is played by the matrix

$$\mathbf{f}_\mathbf{y}(x,\mathbf{y}) = \begin{bmatrix} \dfrac{\partial f_1}{\partial y_1} & \cdots & \dfrac{\partial f_1}{\partial y_m} \\ \vdots & & \vdots \\ \dfrac{\partial f_m}{\partial y_1} & \cdots & \dfrac{\partial f_m}{\partial y_m} \end{bmatrix}$$

We will return to this when needing it in a later section. The analogue of the first order linear equation is

$$\mathbf{y}' = A(x)\mathbf{y} + \mathbf{g}(x), \quad x \geq x_0; \quad \mathbf{y}(x_0) = \mathbf{Y}_0$$

with $A(x)$ a matrix of order $m \times m$.