Residual Correction Example

Solve $A_\lambda x = b$, where A_λ has order n,

$$A_\lambda = \lambda I - K, \qquad K_{i,j} = \frac{1}{n}\kappa(t_i - t_j), \qquad \kappa(u) = \frac{1}{1+u^2}$$

$$t_i = \frac{2i-1}{2n}, \quad b_i = \lambda - \tan^{-1}(1-t_i) - \tan^{-1}(t_i), \quad i = 1, ..., n$$

This is a discretization of the integral equation

$$\lambda f(s) - \int_0^1 \kappa(s-t) f(t)\, dt = \lambda - \tan^{-1}(1-s) - \tan^{-1}(s), \quad 0 \le s \le 1$$

using the midpoint numerical integration rule,

$$\int_0^1 g(t)\, dt \approx \frac{1}{n}\sum_{j=1}^{n} g(t_j)$$

For larger values of n, it can be shown that

$$f(t_i) \approx x_i, \quad i = 1, ..., n$$

Also, in this particular case, $f(t) \equiv 1$.

In the program implementing this approximation scheme:
(1) Read an initial value of λ, calling it λ_0. Solve $A_{\lambda_0} x = b$ and produce the LU factorization of A_{λ_0}.
(2) Read other values of λ, near λ_0, and also read an iteration error tolerance, say $\epsilon > 0$. Use the iteration

$$r^{(m)} = b - A_\lambda x^{(m)}; \qquad x^{(m+1)} = x^{(m)} + \delta^{(m)}$$

$$A_{\lambda_0}\delta^{(m)} = r^{(m)}$$

with the last step solved using the LU factorization of A_{λ_0}.
(3) Calculate and print $\left\| x^{(m+1)} - x^{(m)} \right\|_\infty$ and their successive ratios.

In all cases, we have used $x^{(0)} \equiv 0$, although this is inefficient.