Collocation methods To simplify the presentation, we consider only the BVP

$$y'' = f(x, y, y'), \quad a < x < b$$
$$y(a) = y(b) = 0 \qquad (6.11.34)$$

If the boundary conditions are nonzero, then a simple modification can be made to the unknown y and the differential equation to put the BVP into the form (6.11.34); see problem 54. The collocation methods are much more general than indicated by solving (6.11.34), but the essential ideas are more easily understood in the above context.

We assume the solution Y(x) of (6.11.34) is approximable by a linear combination of n given functions $\psi_1(x), \ldots, \psi_n(x)$:

$$Y(x) \doteq y_n(x) \equiv \sum_{j=1}^{n} c_j \psi_j(x), \quad a \le x \le b \qquad (6.11.35)$$

The functions $\psi_j(x)$ are all assumed to satisfy the boundary conditions,

$$\psi_j(a) = \psi_j(b) = 0, \quad j=1, \ldots, n, \qquad (6.11.36)$$

and thus $y_n(x)$ will also satisfy them, regardless of the choice of constants $\{c_j\}$. The coefficients $c_1, \ldots, c_n$ are determined by forcing the differential equation to be satisfied by $y_n(x)$ at n preselected points in (a,b):

$$y_n''(\xi_i) = f(\xi_i, y_n(\xi_i), y_n'(\xi_i)), \quad i=1,\ldots,n \qquad (6.11.37)$$

with given points

$$a<\xi_1<\xi_2<\ldots<\xi_n<b \qquad (6.11.38)$$

The procedures of defining $y_n(x)$ implicitly thru (6.11.37) is known as *collocation*, and the points $\{\xi_i\}$ are called *collocation points*. Substituting from (6.11.35) into (6.11.37), we obtain

$$\sum_{j=1}^{n} c_j \psi_j''(\xi_i) = f\left[\xi_i, \sum_{j=1}^{n} c_j \psi_j(\xi_i), \sum_{h=1}^{n} c_j \psi_j'(\xi_i)\right], \quad i=1,\ldots,n \qquad (6.11.39)$$

This is a system of n nonlinear equations in the n unknowns $c_1, \ldots, c_n$. In general, this system must be solved numerically.

In choosing a collocation method, we must (1) choose the family of approximating functions $\{\psi_j(x)\}$, (2) choose the collocation points $\{\xi_j\}$, and (3) choose a way to solve the system (6.11.39). With this last step, we will need an initial guess for $\{c_j\}$, and that may be difficult to find. The choice of $\{\psi_j\}$ and $\{\xi_j\}$ can be aided through a theoretical investigation of collocation methods. For a general survey, we refer the reader to Reddien (1979). For the solution of the nonlinear system, see the survey of Deuflhard (1979).

We will describe briefly a particular collocation method that has been implemented as a high quality computer code. Let $m>0$, $h=(b-a)/m$, and define $x_j=a+jh$, $j=0,1,\ldots,m$. Let $k\geq2$, and consider all functions $p(x)$ that satisfy the conditions that

(1) $p(x)$ is continuously differentiable for $a \le x \le b$, (2) $p(a)=p(b)=0$, and (3) on each subinterval $[x_{j-1}, x_j]$, $p(x)$ is a polynomial of degree $< k+2$. We will use these functions as our approximations $y_n(x)$ in (6.11.37). There are a number of ways of writing it in the form (6.11.35), with $n=km$, and we will not consider it here. For $k=2$, such functions are the cubic Hermite functions discussed earlier in §3.7, preceding formula (3.7.9). The functions $\psi_j(x)$ can then be based on the formulas in (3.6.12). The functions $p(x)$ satisfying the above conditions (1)-(3) will be denoted collectively by $P_{k+2,m}$.

For the collocation points, let $\rho_1, \ldots, \rho_k$ be the zeroes of the Legendre polynomial of degree k on [-1,1]. Using them, define

$$\xi_{ij} = \frac{x_{i-1}+x_i}{2} + \frac{1}{2}h\rho_j, \quad 1 \le j \le k, \quad 1 \le i \le m \qquad (6.11.40)$$

This defines $n=mk$ points ξ_{ij}, and these will be the collocation points used in (6.11.37)-(6.11.39).

With this choice for $y_n(x)$ from $P_{k+2,m}$ and $\{\xi_{ij}\}$, and assuming sufficient differentiability and stability in the solvability of the BVP (6.11.34), it can be shown that the solution of (6.11.37) satisfies

$$\underset{a \le x \le b}{\text{Max}} |Y(x) - y_h(x)| = O(h^{k+2})$$

$$\underset{0 \le i \le m}{\text{Max}} |Y(x_i) - y_h(x_i)| = O(h^{2k})$$

These give the same rate of convergence when $k=2$; but for $k>2$, the convergence at the mesh points $\{x_i\}$ is faster than at other points in [a,b]. For a complete discussion of this method, including its practical implementation in a far more general setting than (6.11.34), see Ascher-et.al. (1979).

Other methods and problems There are a number of other methods used for solving boundary value problems; but they have not been developed as high quality general computer programs to the extent of the methods described above. One of these, the reformulation of the BVP as an integral equation, which is then solved numerically, is discussed in Keller (1968, Chap. 4).

There are also many other types of boundary value problems, some containing some type of singular behaviour, that we have not discussed here. For all of these, see the papers in the proceedings of Ascher-Russell (1985), Aziz (1975), Childs-et.al. (1979), and Gladwell-Sayers (1980); also see Keller (1976, Chap. 4) for singular problems. For discussions of software, see Childs-et.al.(1979), Gladwell (1980), and Enright (1985).