A Moderately Ill-conditioned Problem

Consider the linear system

$$\begin{bmatrix} 7 & 10 \\ 5 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 \\ .7 \end{bmatrix}$$

which has the solution

$$x = \begin{bmatrix} 0 \\ .1 \end{bmatrix}$$

Now perturb the data on the right side to obtain the perturbed linear system

$$\begin{bmatrix} 7 & 10 \\ 5 & 7 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} = \begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{bmatrix} = \begin{bmatrix} 1 + \epsilon_1 \\ .7 + \epsilon_2 \end{bmatrix}$$

It has the solution

$$\begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ .1 \end{bmatrix} + \begin{bmatrix} -7\epsilon_1 + 10\epsilon_2 \\ 5\epsilon_1 - 7\epsilon_2 \end{bmatrix}$$

Thus for the data, the relative perturbations are

$$\frac{y_1 - \tilde{y}_1}{y_1} = \epsilon_1, \qquad \frac{y_2 - \tilde{y}_2}{y_2} = \frac{\epsilon_2}{.7}$$

For the answers, the relative perturbation in x_2 is given by

$$\frac{x_2 - \tilde{x}_2}{x_2} = -\frac{5\epsilon_1 + 7\epsilon_2}{0.1} = -50\epsilon_1 + 70\epsilon_2$$

As a particular case, try $[\epsilon_1, \ \epsilon_2] = [.01, \ -.01]$. This yields

$$\begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} = \begin{bmatrix} -.17 \\ .22 \end{bmatrix}$$

Then

$$\frac{x_2 - \tilde{x}_2}{x_2} = -1.2$$

and

$$\left| \frac{x_2 - \tilde{x}_2}{x_2} \right| \div \max \left\{ \left| \frac{y_1 - \tilde{y}_1}{y_1} \right|, \left| \frac{y_2 - \tilde{y}_2}{y_2} \right| \right\} = 84$$

Thus the condition number of the component x_2 satisfies

$$cond(x_2) \geq 84$$

It can be shown to be somewhat larger with other choices of $[\epsilon_1, \ \epsilon_2]$.

An Ill-conditioned Differential Equations Problem

Solve the initial value problem

$$x'(t) = 100x(t) - 100, \qquad 0 < t < 1, \qquad y = x(0) = 1$$

This has the solution

$$x(t) \equiv 1$$

Next consider the perturbed problem

$$\tilde{x}'(t) = 100\tilde{x}(t) - 100, \qquad 0 < t < 1, \qquad \tilde{y} = \tilde{x}(0) = 1 + \epsilon$$

It has the solution

$$\tilde{x}(t) \equiv 1 + \epsilon\, e^{100t}$$

The data has a relative perturbation of

$$\frac{y - \tilde{y}}{y} = -\epsilon$$

But the solution, say at $t = 1$, has the far larger relative perturbation of

$$\frac{x(1) - \tilde{x}(1)}{x(1)} = -\epsilon\, e^{100}$$

Thus for the condition number of $x(1)$,

$$\left|\frac{x(1) - \tilde{x}(1)}{x(1)}\right| \div \left|\frac{y - \tilde{y}}{y}\right| = e^{100}$$

and

$$cond(x(1)) \geq e^{100}$$

This shows that the above initial value problem (1) is very ill-conditioned.