THE NUMERICAL SOLUTION OF A NONLINEAR BOUNDARY INTEGRAL EQUATION ON SMOOTH SURFACES

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Abstract

We study a boundary integral equation method for solving Laplace's equation $\Delta u = 0$ with nonlinear boundary conditions. This nonlinear boundary value problem is reformulated as a nonlinear boundary integral equation, with u on the boundary as the solution being sought. The integral equation is solved numerically by using the collocation method, with piecewise quadratic functions used as approximations to u. Convergence results are given for the cases that (1) the original surface is used, and (2) the surface is approximated by piecewise quadratic interpolation. In addition, we define and analyze a two-grid iteration method for solving the nonlinear system that arises from the discretization of the boundary integral equation. Numerical examples are given; and the paper concludes with a short discussion of the relative cost of different parts of the method.

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1 INTRODUCTION

Consider solving the nonlinear boundary value problem

$$\Delta u(P) = 0, \quad P \in D \tag{1}$$

$$\frac{\partial u(P)}{\partial \mathbf{n}_P} = g(P, u(p)) - f(P), \quad P \in \Gamma = \partial D \tag{2}$$

The region D is to be an open connected region in $\mathbf{R}^3$ with a smooth connected boundary Γ, and $\mathbf{n}_P$ is the interior unit normal at $P \in \Gamma$. We study the numerical solution of a nonlinear boundary integral equation reformulation of this problem, a reformulation that has been studied previously in [23], [8] for planar problems (1)-(2). We seek a solution $u \in C^2(D) \cap C^1(\overline{D})$. Our numerical methods generalize to other problems, for example exterior problems, but such problems are not considered here. The function $g(P, v)$ is assumed to be continuous for $(P, v) \in \Gamma \times \mathbf{R}$, although this can be relaxed. Further assumptions on g are given later.

Using Green's representation formula for harmonic functions, the function u satisfies

$$u(P) = \frac{1}{4\pi} \int_\Gamma \frac{\partial u(Q)}{\partial \mathbf{n}_Q} \frac{dS(Q)}{|P-Q|} - \frac{1}{4\pi} \int_\Gamma u(Q) \frac{\partial}{\partial \mathbf{n}_Q} \left[\frac{1}{|P-Q|} \right] dS(Q), \quad P \in D \tag{3}$$

Letting P tend to a point on Γ, and using the boundary condition (2), we obtain the nonlinear boundary integral equation

$$2\pi\, u(P) - \int_\Gamma u(Q) \frac{\partial}{\partial \mathbf{n}_Q} \left[\frac{1}{|P-Q|} \right] dS(Q) = \int_\Gamma g(Q, u(Q)) \frac{dS(Q)}{|P-Q|} \tag{4}$$

This can be solved for $u(P)$ on Γ. The normal derivative of u can be obtained from (2), and (3) then yields $u(P)$ at all $P \in D$.

We solve (4) by using a boundary element method. A triangulation $\mathcal{T}_N = \{\Delta_{k,N} \mid 1 \le k \le N\}$ is given for Γ, depending on an integer N. Any function $f \in C(\Gamma)$ is approximated on an element $\Delta_{k,N}$ by a function which is polynomial of degree at most d in the parametrization variables for $\Delta_{k,N}$. Let $\mathcal{A}_N$ denote the set of all such functions which are continuous on Γ and are a polynomial of degree at most d on each element $\Delta_{k,N}$ of the triangulation.We assume a solution u_N from $\mathcal{A}_N$ for (4), and we determine u_N by

forcing it to satisfy the integral equation at a selected set of node points on Γ. We note that our results and arguments are given only for the case of quadratic polynomial approximations ($d = 2$); but the ideas involved will generalize to any other degree of polynomial approximation.

Preliminary definitions and assumptions on the triangulation of the surface Γ and on the interpolation scheme being used are given in Section 2. The collocation method is defined and analyzed in Section 3. Practical details of the collocation method (e.g. the calculation of the integrals by numerical integration) are discussed in Section 4, and numerical examples are included. An iteration method for the solution of the associated nonlinear algebraic system is given in Section 5, and numerical examples of the iteration method are given in Section 6. We also discuss the relative cost of the various parts of the solution procedure.

2 PRELIMINARIES

We assume the surface Γ is smooth; and more precisely, at each $P \in \Gamma$, there is a local parametrization which is four times continuously differentiable. In addition, assume Γ can be decomposed as

$$\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_J \tag{5}$$

with each Γ_i the range of a smooth one-to-one mapping on a closed polygonal domain in $\mathbf{R}^2$:

$$F_i : R_i \xrightarrow[onto]{1-1} \Gamma_i, \quad i = 1, \cdots, J, \quad F_i \in C^4(R_i) \tag{6}$$

It is assumed that if two distinct sections Γ_i and Γ_j intersect, then it is only along some portion of their boundaries.

Each R_i is triangulated, say by $\{\widehat{\Delta}_{k,i}\}$, leading to a triangulation of each Γ_i through the application of F_i to each $\widehat{\Delta}_{k,i}$. Collectively these form the triangulation $\mathcal{T}_N = \{\Delta_{k,N}\}$ referred to earlier. It is assumed that if two triangles $\Delta_{j,N}$ and $\Delta_{k,N}$ intersect, then they do so only at a common vertex or along an entire common edge.

We make a further assumption about the refinement process by which the triangular mesh is made finer. An initial triangulation $\{\widehat{\Delta}_{k,i}^{(0)}\}$ is to be given for each R_i, and it is to be refined as follows. For each $\widehat{\Delta}_{k,i}^{(0)}$, connect

the midpoints of the sides of the triangle, to form four new triangles, referred to generically by $\hat{\Delta}_{k,i}^{(1)}$. This new triangulation is refined similarly to form triangles $\hat{\Delta}_{k,i}^{(2)}$, and so on, with each such refinement leading to a new finer triangulation of Γ. For each triangulation $\mathcal{T}_N$, this refinement process leads to a new triangulation with four times as many elements.

As notation, introduce

$$\hat{h} = \max_{i=1,\cdots,J} \left[\max_k \ diameter(\hat{\Delta}_{k,i}) \right]$$

Note that $\hat{h}^2 = O(\frac{1}{N})$; and when using the refinement scheme described above, $\hat{h}$ is halved when N increases by a factor of 4.

Let σ denote the unit simplex,

$$\sigma = \{(s,t) \mid 0 \le s,t,s+t \le 1\}$$

Introduce the nodes

$$q_1 = (0,0), \quad q_2 = (0,1), \quad q_3 = (1,0) \tag{7}$$

Let $\Delta_k \in \mathcal{T}_N$ be associated with the subsection Γ_i. Define a one-to-one mapping of σ onto $\Delta_k \in \mathcal{T}_N$ by

$$m_k(s,t) = F_i(u\,q_1 + t\,q_2 + s\,q_3), \quad (s,t) \in \sigma, \quad u = 1-s-t \tag{8}$$

Integrals over Δ_k are evaluated as integrals over σ:

$$\int_{\Delta_k} g(Q)\,dS = \int_\sigma g(m_k(s,t))\,|D_s m_k \times D_t m_k|\,d\sigma \tag{9}$$

with D_s and D_t referring to derivatives with respect to s and t, respectively. Later in Section 4, we discuss the numerical evaluation of such integrals.

To approximate u, we use functions which are piecewise quadratic on the triangulation. More precisely, the approximants are to be continuous on S and they are to be quadratic on each Δ_k in the parametrization variables from $\hat{\Delta}_k$. We begin by defining such functions and the associated interpolation on the reference element σ. Introduce the nodes

$$q_4 = (0,0.5), \quad q_5 = (0.5,0.5), \quad q_6 = (0.5,0) \tag{10}$$

Introduce the quadratic interpolation basis functions:

$$\begin{array}{lllllllll} l_1(s,t) & = & u(2u-1), & l_2(s,t) & = & t(2t-1), & l_3(s,t) & = & s(2s-1) \\ l_4(s,t) & = & 4tu, & l_5(s,t) & = & 4st, & l_6(s,t) & = & 4su \end{array} \tag{11}$$

in which $u = 1 - s - t$. These polynomials satisfy

$$l_i(q_j) = \delta_{ij}$$

For a function $f \in C(\sigma)$, the polynomial

$$\sum_{j=1}^{6} f(q_j)\, l_j(s,t) \tag{12}$$

interpolates the function f at the nodes $\{q_j\}$; and among polynomials of degree at most two, it is unique.

Now consider interpolation to a function $g \in C(S)$. Define the interpolant on Δ_k by

$$\mathcal{P}_N g(m_k(s,t)) = \sum_{j=1}^{6} g(m_k(q_j))\, l_j(s,t), \qquad (s,t) \in \sigma \tag{13}$$

This defines a continuous function on Γ, which we refer to as a *piecewise quadratic interpolation function.* The collection of all such piecewise quadratic functions gives the approximating subspace $\mathcal{A}_N$.

If $f \in C(\Gamma)$, then it is straightforward to show that $\|f - \mathcal{P}_N f\|_\infty \to 0$ as $N \to \infty$. In addition, if $f \in C^4(\Gamma)$, then

$$\|f - \mathcal{P}_N f\|_\infty = O(\hat{h}^3) \tag{14}$$

Finally, it is also straightforward to show

$$\|\mathcal{P}_N\| = \frac{5}{3}$$

The points

$$v_{j,k} = m_k(q_j), \qquad j = 1, \cdots, 6, \qquad k = 1, \cdots, N$$

lay on Γ, and we refer to them as the nodes of the triangulation. The nodes $v_{1,k}$, $v_{2,k}$, $v_{3,k}$ are the vertices of Δ_k, and the nodes $v_{4,k}$, $v_{5,k}$, $v_{6,k}$ can be

regarded as approximate midpoints of the sides of Δ_k. Collectively, the number of such nodes is

$$N_v = 2(N+1)$$

We also refer to these nodes as

$$\mathcal{V}_N = \{v_1, \cdots, v_{N_v}\}$$

when we are not considering the triangular element to which they belong.

When evaluating surface integrals, we need an explicitly differentiable parametrization of the surface. More precisely,

$$\int_{\Delta_k} f(Q)\, dS = \int_\sigma f(m_k(s,t))\, |(D_s m_k \times D_t m_k)(s,t)|\, d\sigma \tag{15}$$

The derivatives $D_s m_k$ and $D_t m_k$ involve the derivatives of F_i, as in (12). For polyhedral surfaces S, there is no problem in obtaining these derivatives. But for surfaces which are curved, this can be a major inconvenience in using high order boundary element methods. For that reason, we introduce approximations of the surface, and we use their derivatives to approximate those used in (15).

For a triangle Δ_k, define

$$\tilde{m}_k(s,t) = \sum_{j=1}^{6} m_k(q_j)\, l_j(s,t), \qquad (s,t) \in \sigma \tag{16}$$

We refer to the image of $\tilde{m}_k$ as $\tilde{\Delta}_k$. The triangles Δ_k and $\tilde{\Delta}_k$ agree at the nodes $v_{1,k}, \cdots, v_{6,k}$. The union of all such $\tilde{\Delta}_k$ is denoted by $\tilde{\Gamma} \equiv \tilde{\Gamma}_N$; and this is a continuous and piecewise smooth surface.

The integral in (15) becomes

$$\int_{\Delta_k} f(Q)\, dS \doteq \int_\sigma f(\tilde{m}_k(s,t))\, |(D_s \tilde{m}_k \times D_t \tilde{m}_k)(s,t)|\, d\sigma \tag{17}$$

The function f is now being evaluated at points off of Γ, but that is not a major practical or theoretical difficulty. We can always extend the function f to a neighborhood of S in such a way as to maintain its differentiability, subject to Γ being sufficiently differentiable. In practice, we approximate (17) for cases in which the extension of f is known explicitly. Alternatively, when numerical quadrature is used for a general function f, our quadrature

formulas often evaluate $f(\tilde{m}_k(s,t))$ only at the points $(s,t) = q_1, \cdots, q_6$; and in this case, f is being evaluated at points on the original surface. Evaluating $D_s\tilde{m}_k \times D_t\tilde{m}_k$ is relatively straightforward and can be incorporated into a general package for triangulating and approximating S.

Detailed error analyses for the above approximations and for various quadrature schemes are given in Chien [12], [13]; and for their use in solving boundary integral equations, see Atkinson and Chien [9]. We will refer to these papers when necessary, to avoid duplication of complicated arguments.

Only the use of piecewise quadratic interpolation in solving the boundary integral equation (4) is analyzed in this paper; but the methods of analysis are not limited to this case. Chien [13] discusses the use of piecewise quadratic interpolation for integral equations with smooth kernel functions; and he shows how the central ideas extend to other degrees of piecewise polynomial interpolation. We have concentrated our attention on the use of piecewise quadratic interpolation because (1) it is simpler to write about one particular degree of approximation, (2) we wanted to analyze and program something significantly more complicated than the more well-known panel methods, and (3) the use of piecewise quadratic approximations is important in its own right.

3 THE COLLOCATION METHOD

The integral equation (4) can be written abstractly as

$$u = \mathcal{K}u + \mathcal{S}\mathcal{G}(u) \equiv \mathcal{L}(u) \tag{18}$$

In this equation, $\mathcal{K}$ is the double layer potential operator

$$\mathcal{K}\rho(P) = \frac{1}{2\pi}\int_\Gamma \rho(Q)\frac{\partial}{\partial \mathbf{n}_Q}\left[\frac{1}{|P-Q|}\right] dS(Q), \quad P \in \Gamma \tag{19}$$

$\mathcal{S}$ is the single layer potential operator

$$\mathcal{S}\rho(P) = \frac{1}{2\pi}\int_\Gamma \frac{\rho(Q)}{|P-Q|}\, dS(Q), \quad P \in \Gamma \tag{20}$$

and $\mathcal{G}$ is the Nemytskii operator

$$(\mathcal{G}(\rho))(P) = g(P, \rho(P)), \quad P \in \Gamma \tag{21}$$

In (18)-(20), ρ is an arbitrary element from the function space in which (18) is to be analyzed. We use $C(\Gamma)$ with the uniform norm as the space for all of our error analyses.

With Γ a smooth surface, as assumed in Section 2, the operators $\mathcal{K}$ and $\mathcal{S}$ are compact linear operators on $C(\Gamma)$ into $C(\Gamma)$. When Γ is only piecewise smooth, $\mathcal{K}$ is no longer compact; and it acts in some ways like the Dirac delta function. This lack of compactness in $\mathcal{K}$ for a piecewise smooth boundary Γ leads to non-trivial changes in both the numerical methods for (18) and their error analysis. For this reason, we consider only the case of a smooth boundary Γ in this paper.

With the assumption of continuity of $g(P,v)$ for $(P,v) \in \Gamma \times \mathbf{R}$, it follows easily that $\mathcal{G}$ maps bounded subsets of $C(\Gamma)$ into bounded subsets of $C(\Gamma)$. We make the further assumption that

$$g_v(P,v) = \frac{\partial g}{\partial v} \quad \text{and} \quad g_{vv}(P,v) = \frac{\partial^2 g}{\partial v^2}$$

are continuous for $(P,v) \in \Gamma \times \mathbf{R}$. This assures the existence of the Frechet derivative $\mathcal{G}'(\rho)$ as a bounded linear operator on $C(\Gamma)$, for all $\rho \in C(\Gamma)$; and moreover, $\mathcal{G}'(\rho)$ is continuous in ρ with respect to the operator norm on $C(\Gamma)$. These assumptions on g can be relaxed to ones which are local with respect to a solution u^* of (18); but the present assumptions simplify our presentation, without making any major difference to the final results.

With the above assumptions on Γ and g, the nonlinear operator $\mathcal{L}$ is completely continuous on $C(\Gamma)$ to $C(\Gamma)$. Moreover, for each $u \in C(\Gamma)$, $\mathcal{L}'(u)$ is a compact linear operator on $C(\Gamma)$ to $C(\Gamma)$;and the mappings $u \longmapsto \mathcal{L}'(u)$ is continuous as a mapping from $C(\Gamma)$ to the space of bounded linear operators on $C(\Gamma)$ to $C(\Gamma)$. The classical tools and results for the analysis of (18) are given in Krasnoselskii [22], and we will refer to them as needed.

Using the interpolatory projection operator $\mathcal{P}_N$ of Section 2, we approximate (18) by

$$u_N = \mathcal{P}_N \mathcal{L}(u_N) \tag{22}$$

We henceforth let u^* denote the desired solution of (18), and we let u_N^* denote the desired solution of (22), provided such a solution exists. The method in (22) is a collocation method, and solving it reduces to the solution of a finite system of nonlinear algebraic equations. Later in this section, we consider the actual system being solved.

The abstract error analysis for (22) was first given by Krasnoselskii [22, p. 169]. In [1] and [11], these results were extended, and we will refer to these papers in our error analysis. For this analysis, we need to assume that 1 is not an eigenvalue of $\mathcal{L}'(u^*)$, which is equivalent to assuming

$$I - \mathcal{L}'(u^*) : C(\Gamma) \underset{onto}{\overset{1-1}{\to}} C(\Gamma) \tag{23}$$

This assumption is the generalization of assuming the Jacobian matrix is nonsingular when solving a finite algebraic system of equations; and for a root u^* of a single equation $f(u) = 0$ of a real variable u, this assumption is equivalent to assuming the root u^* is simple. Under suitable assumptions on g, it can be shown that (18) has a unique solution u^* and that 1 is not an eigenvalue of $\mathcal{L}'(u^*)$; but we do not discuss such results here.

Theorem 1 *Assume Γ is a four times continuously differentiable surface. Assume $g(P, v)$ is twice continuously differentiable with respect to v, for $(P, v) \in \Gamma \times \mathbf{R}$. Moreover, assume $\mathcal{G}$ maps $C^r(\Gamma)$ into $C^r(\Gamma)$ for $r \leq 4$. Let u^* denote an isolated solution of (18), and assume $u^* \in C^4(\Gamma)$. Further, assume 1 is not an eigenvalue of $\mathcal{L}'(u^*)$. Then for some $\epsilon > 0$ and for all sufficiently large N, say $N \geq N_0$, the equation (22) has a solution u_N^* which is unique within the neighborhood $\{u \mid \|u^* - u\|_\infty \leq \epsilon\}$. Moreover,*

$$\|u^* - u_N^*\|_\infty \leq c \|u^* - \mathcal{P}_N u^*\|_\infty , \quad N \geq N_0 \tag{24}$$

for some $c > 0$. Thus

$$\|u^* - u_N^*\|_\infty = O(\hat{h}^3) \tag{25}$$

Proof. The proof is omitted, as it is a straightforward repetition of derivations given in [1] and [11]. However, we need certain results obtained in those proofs, and we state those here. There is an open neighborhood Ω of u^* and an integer N_0 for which the following is true.

1. The solution u^* is the unique solution within Ω of equation (18).
2. The equation (22) has a unique solution u_N^* within Ω, $N \geq N_0$.
3. The Frechet derivatives $\mathcal{L}'(u)$, $\mathcal{L}''(u)$ exist and are uniformly bounded for $u \in \Omega$. Note that his implies the uniform boundedness of the first and second derivatives of $\mathcal{P}_N \mathcal{L}(u)$, for $N \geq 1$ and $u \in \Omega$.

4. For all $u, v \in \Omega$,

$$\|\mathcal{L}'(u) - \mathcal{L}'(v)\| \leq b \|u - v\|_\infty, \quad b = \max_{w \in \Omega} \|\mathcal{L}''(w)\| \tag{26}$$

5. The inverses $[I - \mathcal{L}'(u)]^{-1}$ and $[I - \mathcal{P}_N \mathcal{L}'(u)]^{-1}$ exist for $N \geq N_0$ and they are uniformly bounded, say by $c > 0$. Moreover,

$$\left\| [I - \mathcal{L}'(u)]^{-1} v - [I - \mathcal{P}_N \mathcal{L}'(u)]^{-1} v \right\|_\infty \leq c \left\| (I - \mathcal{P}_N) \mathcal{L}'(u) [I - \mathcal{L}'(u)]^{-1} v \right\|_\infty \tag{27}$$

for all $N \geq N_0$ and $u \in \Omega$. □

3.1 The finite algebraic system

The equation (22) is equivalent to a finite system of algebraic equations. Using the notation of Section 2, write

$$u_N^*(m_k(s,t)) = \sum_{j=1}^{6} u_N^*(v_{j,k}) l_j(s,t), \quad (s,t) \in \sigma, \quad k = 1, \cdots, N \tag{28}$$

to define $u_N^*(P)$, for $P \in \Delta_k$. We substitute this into the equation (3) and collocate at the nodes $v_1, \cdots, v_{N_v}$ of the triangulation. This leads to the nonlinear system

$$\mathbf{u}_N = L_N(\mathbf{u}_N), \quad \mathbf{u}_N \in \mathbf{R}^{N_v} \tag{29}$$

with solution $\mathbf{u}_N^*$, the restriction of u_N^* to the collocation nodes. The vectors $\mathbf{u} \in \mathbf{R}^{N_v}$ will be treated as discrete functions with domain $\{v_i \mid i = 1, \cdots, N_v\}$, and the components of $\mathbf{u}$ will be denoted by $\mathbf{u}(v_i)$, $i = 1, \cdots, N_v$. The nonlinear operator $L_N : \mathbf{R}^{N_v} \to \mathbf{R}^{N_v}$ is defined by

$$L_N(\mathbf{u})(v_i) = \frac{1}{2\pi} \sum_{k=1}^{N} \int_{\Delta_k} (\mathcal{E}_N \mathbf{u})(Q) \frac{\partial}{\partial \mathbf{n}_Q} \left[\frac{1}{|v_i - Q|} \right] dS_Q$$

$$+ \frac{1}{2\pi} \sum_{k=1}^{N} \int_{\Delta_k} g(Q, (\mathcal{E}_N \mathbf{u})(Q)) \frac{1}{|v_i - Q|} dS_Q, \quad \mathbf{u} \in \mathbf{R}^{N_v}$$

with the *extension* or *prolongation* operator $\mathcal{E}_N$ defined by

$$(\mathcal{E}_N \mathbf{u})(m_k(s,t)) = \sum_{j=1}^{6} \mathbf{u}(v_{jk}) l_j(s,t), \quad (s,t) \in \sigma, \quad k = 1, \cdots, N \tag{30}$$

Using the parametrization $m_k : \sigma \overset{1-1}{\underset{onto}{\to}} \Delta_k$ of Section 2, we can write

$$L_N(\mathbf{u})(v_i) = \frac{1}{2\pi} \sum_{k=1}^{N} \sum_{j=1}^{6} \mathbf{u}(v_{jk}) \int_\sigma l_j(s,t) \frac{\partial}{\partial \mathbf{n}_Q} \left[\frac{1}{|v_i - m_k(s,t)|} \right] |D_s m_k \times D_t m_k| \, d\sigma \tag{31}$$

$$+ \frac{1}{2\pi} \sum_{k=1}^{N} \int_{\Delta_k} g(m_k(s,t), \sum_{j=1}^{6} \mathbf{u}(v_{jk}) l_j(s,t)) \frac{1}{|v_i - m_k(s,t)|} |D_s m_k \times D_t m_k| \, d\sigma$$

for arbitrary $\mathbf{u} \in \mathbf{R}^{N_v}$. Bounds on $L_N(\mathbf{u})$ and its derivatives follow from corresponding bounds for $\mathcal{P}_N \mathcal{L}$ and its derivatives. The bounds for vectors in $\mathbf{R}^{N_v}$ and matrices on it are based on the vector norm $\|\cdot\|_\infty$ and the associated matrix row norm.

We approximate these integrals further by replacing $m_k(s,t)$ by the interpolating approximation $\widetilde{m}_k(s,t)$ of (16). This yields the new nonlinear operator

$$\widetilde{L}_N(\mathbf{u})(v_i) = \frac{1}{2\pi} \sum_{k=1}^{N} \sum_{j=1}^{6} \mathbf{u}(v_{jk}) \int_\sigma l_j(s,t) \frac{\partial}{\partial \mathbf{n}_Q} \left[\frac{1}{|v_i - \widetilde{m}_k(s,t)|} \right] |D_s \widetilde{m}_k \times D_t \widetilde{m}_k| \, d\sigma \tag{32}$$

$$+ \frac{1}{2\pi} \sum_{k=1}^{N} \int_{\Delta_k} g(\widetilde{m}_k(s,t), \sum_{j=1}^{6} \mathbf{u}(v_{jk}) l_j(s,t)) \frac{1}{|v_i - \widetilde{m}_k(s,t)|} |D_s \widetilde{m}_k \times D_t \widetilde{m}_k| \, d\sigma$$

for $\mathbf{u} \in \mathbf{R}^{N_v}$. We actually solve the nonlinear system

$$\mathbf{u}_N = \widetilde{L}_N(\mathbf{u}_N), \quad \mathbf{u}_N \in \mathbf{R}^{N_v} \tag{33}$$

and its solution is denoted by $\widetilde{\mathbf{u}}_N^*$ (provided it exists). The integrals in (32) must still be evaluated numerically, but we defer the discussion of this until Section 4.

Let $\mathcal{R}_N : C(\Gamma) \overset{onto}{\to} \mathbf{R}^{N_v}$ be the *restriction* operator, so that $\mathcal{R}_N u$ is the restriction of u to the collocation nodes $\{v_i\}$.

Theorem 2 *Make the same assumptions as in Theorem 1. Then there is an integer $N_1 \geq N_0$ and an $\epsilon > 0$ for which we have the following. For all $N \geq N_1$, (33) has a solution $\widetilde{\mathbf{u}}_N^*$, and it is unique within an ϵ-neighborhood of $\mathcal{R}_N u^*$ in $\mathbf{R}^{N_v}$. Moreover,*

$$\max_i |u^*(v_i) - \widetilde{\mathbf{u}}_N^*(v_i)| \leq c\hat{h}^2 \tag{34}$$

Proof. We reformulate the equation (33) as an equivalent equation to which the Contractive Mapping Theorem (cf. [18], p. 627) can be applied, uniformly with respect to N. For notation, c will be treated as a generic constant, uniform in N for N sufficiently large.

In (33), expand $\widetilde{L}_N(\mathbf{u}_N)$ in a linear Taylor series about $\mathbf{u}_N^*$, with remainder:

$$\widetilde{L}_N(\mathbf{u}_N) = \widetilde{L}_N(\mathbf{u}_N^*) + \widetilde{L}'_N(\mathbf{u}_N^*)(\mathbf{u}_N - \mathbf{u}_N^*) + R_N(\mathbf{u}_N^*; \mathbf{u}_N - \mathbf{u}_N^*) \tag{35}$$

with

$$R_N(\mathbf{u}_N^*; \delta) = O(\|\delta\|_\infty^2) \tag{36}$$

Let $\delta = \mathbf{u}_N - \mathbf{u}_N^*$, subtract $\mathbf{u}_N^* = L_N(\mathbf{u}_N^*)$ from (35), and rewrite the result as

$$[I - \widetilde{L}'_N(\mathbf{u}_N^*)]\delta = \widetilde{L}_N(\mathbf{u}_N^*) - L_N(\mathbf{u}_N^*) + R_N(\mathbf{u}_N^*; \delta) \tag{37}$$

From the uniform boundedness of $[I - \mathcal{P}_N\mathcal{L}'(u)]^{-1}$ for $N \geq N_0$ and $u \in B$, it is relatively straightforward to show that the same is true of $[I - L'_N(\mathbf{u}_N^*)]^{-1}$. In addition, the matrix $\widetilde{L}'_N(\mathbf{u}_N^*)$ is obtained from $L'_N(\mathbf{u}_N^*)$ by replacing m_k by $\widetilde{m}_k$, as in (32) above. From a proof very similar to that of [9, Theorem 5.2], we can show

$$\left\| \widetilde{L}'_N(\mathbf{u}_N^*) - L'_N(\mathbf{u}_N^*) \right\| \leq c\widehat{h}^2, \quad N \geq N_0 \tag{38}$$

It is then straightforward to show that $[I - \widetilde{L}'_N(\mathbf{u}_N^*)]^{-1}$ exists and is uniformly bounded for for sufficiently large N.

The equation (37) can then be written

$$\delta = [I - \widetilde{L}'_N(\mathbf{u}_N^*)]^{-1} \left\{ \left[\widetilde{L}_N(\mathbf{u}_N^*) - L_N(\mathbf{u}_N^*) \right] + R_N(\mathbf{u}_N^*; \delta) \right\} \tag{39}$$

Write this as

$$\delta = \mathcal{M}_N(\delta)$$

To apply the Contractive Mapping Theorem, we must show

(A1) $\mathcal{M}_N$ maps some ball $B(\epsilon) \equiv \{\mathbf{u} \mid \|\mathbf{u}\|_\infty \leq \epsilon\}$ about zero into itself, uniformly with respect to N; and

(A2) $\mathcal{M}_N$ is contractive on $B(\epsilon)$, uniformly with respect to N.

From (26), we can show for the collocation discretization that

$$\left\|L_N(\mathbf{u}_N^*) - L_N(\mathcal{R}_N u^*)\right\|_\infty \le b_1 \left\|\mathbf{u}_N^* - \mathcal{R}_N u^*\right\|_\infty = O(\widehat{h}^3) \tag{40}$$

$$\left\|\widetilde{L}_N(\mathbf{u}_N^*) - \widetilde{L}_N(\mathcal{R}_N u^*)\right\|_\infty \le b_1 \left\|\mathbf{u}_N^* - \mathcal{R}_N u^*\right\|_\infty = O(\widehat{h}^3)$$

for a suitable constant $b_1 > 0$. Consequently

$$\begin{aligned} \left\|\widetilde{L}_N(\mathbf{u}_N^*) - L_N(\mathbf{u}_N^*)\right\|_\infty &\le 2b_1 \left\|\mathbf{u}_N^* - \mathcal{R}_N u^*\right\|_\infty + \left\|L_N(\mathcal{R}_N u^*) - \widetilde{L}_N(\mathcal{R}_N u^*)\right\|_\infty \\ &= O(\widehat{h}^2) \end{aligned} \tag{41}$$

Combining this with (36) shows A1, provided ϵ is chosen sufficiently small.

To show $\mathcal{M}_N$ is contractive, we need the following result for arbitrary twice-differentiable nonlinear operators $\mathcal{F}$ on an open set S, with $u^* \in S$:

$$\mathcal{F}(u) = \mathcal{F}(u^*) + \mathcal{F}'(u^*)(u - u^*) + \mathcal{R}(u^*; u - u^*)$$

For η sufficiently small, and for $\|\delta_1\|\,,\ \|\delta_2\| \le \eta$,

$$\|\mathcal{R}(u^*;\delta_1) - \mathcal{R}(u^*;\delta_2)\| \le \alpha(u^*;\delta_1)\,\|\delta_1 - \delta_2\|$$

$$\alpha(u^*;\delta_1) = \left[\frac{1}{2}\,\|\delta_1 - \delta_2\| + \|\delta_1\|\right] \max_{\|\delta\| \le \eta} \|\mathcal{F}''(u^* + \delta)\|$$

As this is a standard result, we omit the proof. The constant $\alpha(u^*;\delta_1)$ can be made arbitrarily small by choosing η suitably small; and using it, the mapping $\mathcal{R}(u^*;\delta)$ can be shown to be contractive with respect to δ.

We apply this to the operators $R_N(\mathbf{u}_N^*;\delta)$ of (39). Using the results given above, including the uniform boundedness of $\mathcal{P}_N\mathcal{L}''(u)$ for $u \in B$ and $N \ge N_0$, we obtain contractiveness for both $R_N(\mathbf{u}_N^*;\delta)$ and $\mathcal{M}_N(\delta)$, uniformly in N.

Using the Contractive Mapping Theorem cited earlier, we obtain the existence of the solution $\widetilde{\mathbf{u}}_N^*$ of (33), unique within some ϵ-neighborhood as asserted. Moreover,

$$\left\|\widetilde{\mathbf{u}}_N^* - \mathbf{u}_N^*\right\|_\infty \le c\left\|\widetilde{L}_N(\mathbf{u}_N^*) - L_N(\mathbf{u}_N^*)\right\|_\infty = O(\widehat{h}^2)$$

with the latter also based on (41). □

The empirical results we obtain in Section 4 seem to imply the error is $O(\widehat{h}^4)$. We hypothesize that this is due to two effects. First, we believe the result (34) should be $O(\widehat{h}^3)$; and to prove this, we believe the result (38) can be improved to $O(\widehat{h}^3)$, although we have not yet been able to do so. Second, we make an additional modification to the nonlinear system (33), and it further improves the convergence to $O(\widehat{h}^4)$.

4 NUMERICAL EXAMPLES

The collocation method of §3 was implemented with the aid of a package of programs which we have designed for solving boundary element methods on smooth and piecewise smooth surfaces. The package was first described in [4], [5]; and it has since been updated and improved in several ways. We expect to make it available for general use in the near future, with an accompanying user's manual.

There are two additional aspects of the practical implementation of the collocation method analyzed in §3: the numerical approximation of the various integrals in (32), and the iterative solution of the nonlinear system (33). We discuss now the calculation of the integrals, and the iteration method is taken up in §5 and §6. To avoid unnecessary repetitions in our evaluations of integrals, note that the bulk of the work in the evaluation of (32) at a particular $\mathbf{u} \in \mathbf{R}^{N_v}$ is in the numerical integration of the single and double layer potentials for the quadratic basis density functions $l_j(s,t)$ of (11). Preceding the iterative solution of (33), we set up matrices for these collocation integrals for the linear single and double layer operators $\mathcal{S}$ and $\mathcal{K}$.

We give only a sketch of the ideas we use in evaluating the integrals in (32), and refer to [9] for a more complete discussion. Consider the evaluation of a canonical integral

$$\int_\sigma \kappa(v_i, \widetilde{m}_K(s,t)) l_j(s,t)\, |D_s \widetilde{m}_K \times D_t \widetilde{m}_K|\, d\sigma \tag{42}$$

in which $\kappa(P,Q)$ denotes the kernel function of either the single layer or double layer operator.

When $v_i \in \Delta_K$, the integrand is singular; and for purposes of explanation, assume, $v_i = m_K(0,0)$. Then we use the transformation

$$s = (1-y)x, \qquad t = yx, \qquad 0 \le x, y \le 1$$

With this, the singularity in the integrand is removed entirely, including in the derivatives of the integrand. We then apply a repeated standard Gauss-Legendre quadrature, say with N_g nodes in each of the variables x and y. Generally we have found $N_g \leq 10$ to be quite sufficient. This transformation has been used previously; for example, see [5], [14], [17], and [24]. The latter paper of Schwab and Wendland [24] contains an extensive analysis of this transformation, including extensions of its use to other types of problems. Another discussion of numerical integration for boundary element methods is given in [15].

For $v_i \notin \Delta_K$, the integrand in (42) is analytic; but it is increasingly peaked as the distance between v_i and Δ_K decreases. Our integration of (42) is based on the formula T2:5-1 of Stroud [25], p. 314:

$$\int_\sigma g(s,t)d\sigma \approx \sum_{j=1}^{7} w_j g(s_j, t_j) \tag{43}$$

It has degree of precision five. Note that the number of integrals in which $v_i \in \Delta_K$ is of order $O(N_v)$, whereas the number of integrals in which $v_i \notin \Delta_K$ is of order $O(N_v^2)$. Thus the total cost of evaluating the latter integrals is likely to be more time-consuming as N increases.

Let

$$\delta_N = \max_K \; diameter(\Delta_K)$$

Let an integer $N_d \geq 0$ be given by the user. If $v_i \notin \Delta_K$ and if

$$dist(v_i, \Delta_K) \leq \delta_N$$

then integrate (42) using N_d levels of subdivision of σ [thus subdividing σ into 4^{N_d} congruent subtriangles, with (43) applied to the integral over each of the corresponding subintegrals]. If $v_i \notin \Delta_K$ and

$$\delta_N < dist(v_i, \Delta_K) \leq 2\delta_N,$$

then integrate (42) using (43) with $\max\{N_d - 1, 0\}$ levels of subdivision of σ. If $v_i \notin \Delta_K$ and

$$2\delta_N < dist(v_i, \Delta_K) \leq 3\delta_N,$$

then integrate (42) using (43) with $\max\{N_d - 2, 0\}$ levels of subdivision of σ. Continue with this in the obvious way. Generally we have found that it is

more than adequate to increase N_d by 1 when N is increased to $4N$. That would mean that when using $N = 8, 32, 128, 512$, that we would use $N_d = 0, 1, 2, 3$. However we have found it quite satisfactory to use $N_d = 0, 1, 2, 2$ or even $N_d = 0, 1, 1, 2$ for the above choices of N.

There is one other important factor in defining the discretization of the double layer integral operator. For the double layer integral operator, we have the identity

$$\int_\Gamma \frac{\partial}{\partial \mathbf{n}_P} \left[\frac{1}{|P-Q|} \right] dS_Q = -2\pi, \quad P \in \Gamma \tag{44}$$

After performing the quadratures described above, we add a "correction term" to the diagonal element of each row of the matrix for the discretization of the double layer, to force the sum of the elements for each row to equal -2π. This is discussed in [9], in which an illustration is given of the usefulness of the above "correction" in increasing the order of convergence.

4.1 The Surfaces

Two surfaces were used in our experiments. Surface #1 (denoted by $S\#1$) was the ellipsoid

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{x}{c}\right)^2 = 1$$

In Table 1 given below, $(a, b, c) = (2, 1.5, 1)$.

The ellipsoid is convex and symmetric. For that reason, we have devised and used a surface which is neither convex nor symmetric. Surface #2 ($S\#2$) is defined by

$$(x, y, z) = \rho(\xi, \eta, \zeta)(A\xi, B\eta, C\zeta), \quad \xi^2 + \eta^2 + \zeta^2 = 1$$

with

$$\rho(\xi, \eta, \zeta) = 1 - \frac{1}{\alpha}\left[(\xi - 0.1)^2 + 2(\eta - 0.1)^2 - 3(\zeta - 0.1)^2\right]$$

and $A, B, C > 0$, $\alpha \geq 5$. The case we use here is $\alpha = 10$ and $(A, B, C) = (2, 2, 1)$. Figure 1 gives the intersections of the surface Γ with the vertical planes containing the z-axis and intersecting the xz-plane at angles of $\phi = 0, \pi/4, \pi/2$. Experiments were done with other choices of α and (A, B, C), corresponding to surfaces with a more pronounced lack of symmetry and

Table 1: Maximum errors for $\widetilde{\mathbf{u}}^*_{1N}$ and $\widetilde{\mathbf{u}}^*_{2N}$ on $S\#1$.

N	$\|\mathcal{R}_N u - \widetilde{\mathbf{u}}^*_{\mathbf{1N}}\|_\infty$	$Order$	$\|\mathcal{R}_N u - \widetilde{\mathbf{u}}^*_{2N}\|_\infty$	$Order$
8	3.62E−1		8.14E−1	
32	5.72E−2	2.66	2.53E−1	1.69
128	4.48E−3	3.67	2.06E−2	3.61
512	2.88E−4	3.96	1.40E−3	3.88

convexity. But in order to obtain error results with some regularity in the asymptotic behavior, we chose the parameters given above, giving the surface illustrated in Figure 1.

4.2 Convergence Results

The problem (1)-(2) was solved with known values of the solution $u(P)$ and

$$g(P, v) = v + \sin(v)$$

The function $f(P)$ was then determined by

$$f(P) = g(P, u(P)) - \frac{\partial u(P)}{\partial \mathbf{n}_P}, \quad P \in \Gamma$$

The true solutions chosen were

$$u_1(x, y, z) = x^2 + y^2 - 2z^2$$

$$u_2(x, y, z) = e^x \cos(y) + e^z \sin(x)$$

These are quite well-behaved functions, although u_2 does vary a great deal over the surface Γ.

The maximum errors for the collocation solutions at the node points for surfaces $S\#1$ and $S\#2$ are given in Tables 1 and 2, respectively. In the tables, the column labeled *Order* gives the empirical order of convergence of the collocation solutions, i.e. the error is $O(\widehat{h}^{Order})$.

Looking at these numerical results, the empirical order for $S\#1$ appears to be approaching 4, and this is much better than the order of 2 predicted

Table 2: Maximum errors for $\widetilde{\mathbf{u}}^*_{1N}$ and $\widetilde{\mathbf{u}}^*_{2N}$ on $S\#2$.

N	$\|\mathcal{R}_N u - \widetilde{\mathbf{u}}^*_{\mathbf{1N}}\|_\infty$	$Order$	$\|\mathcal{R}_N u - \widetilde{\mathbf{u}}^*_{\mathbf{2N}}\|_\infty$	$Order$
8	1.68E+0		2.71E−1	
32	3.20E−1	2.40	2.05E−1	0.41
128	3.84E−2	3.06	3.92E−2	2.38
512	4.00E−3	3.26	3.76E−3	3.38

by (34) in Theorem 2. For $S\#2$, the empirical order is also increasing; and we believe that as N increases further, the empirical order will also approach 4. This is also based on looking at the error at individual node points, in which the empirical order is more clearly approaching 4. As in [9] for the case of linear boundary integral equations, it appears likely that the increase in order is due to two effects. First, the result (34) in Theorem 2 should probably be $O(\widehat{h}^3)$, based on other numerical experiments. Second, the use of a "correction" based on (44) appears to add one additional power to the order of convergence. We do not yet have a rigorous explanation, but our many computations lead us to this conclusion.

5 THE ITERATION METHOD

To solve the nonlinear system (33), we use a modified Newton's method. We begin by supposing that the system to be solved is

$$\mathbf{u}_M = \widetilde{L}_M(\mathbf{u}_M), \qquad \mathbf{u}_M \in \mathbf{R}^{M_v} \tag{45}$$

for an increasing sequence of values M; and we denote its solution by $\widetilde{\mathbf{u}}^*_M$. Our modification of Newton's method will use a two-grid approximation to $\left[I - \widetilde{L}'_M(\mathbf{u}_M)\right]^{-1}$, based on an inverse $\left[I - \widetilde{L}'_N(\mathbf{w})\right]^{-1}$ for some $N < M$ and some point $\mathbf{w} \in \mathbf{R}^{N_v}$ which is close to $\mathcal{R}_N u^*$. The iteration method is very similar to that first given in [2]; but now we use extension and restriction operators to move between spaces of different finite dimension, whereas Nyström interpolation was used in [2]. This is also similar to what is used in Hackbusch [16] for the two-grid approximation of linear integral operators. For other discussions of the use of two-grid iteration methods of the type used

here, see [19], [20], [21]. In the following we initially work with the exact surface Γ; and later we replace it with the approximate surface $\widetilde{\Gamma}_M$.

Some additional notation is needed. Let matrices S_M and K_M be defined by

$$S_M\delta(v_i) = \frac{1}{2\pi}\sum_{k=1}^{M}\sum_{j=1}^{6}\delta(v_{jk})\int_\sigma l_j(s,t)\frac{1}{|v_i - m_k(s,t)|}\,|D_s m_k \times D_t m_k|\;d\sigma$$

and

$$K_M\delta(v_i) = \frac{1}{2\pi}\sum_{k=1}^{M}\sum_{j=1}^{6}\delta(v_{jk})\int_\sigma l_j(s,t)\frac{\partial}{\partial \mathbf{n}_Q}\left[\frac{1}{|v_i - m_k(s,t)|}\right]|D_s m_k \times D_t m_k|\;d\sigma$$

with $Q = m_k(s,t)$, for $i = 1,\cdots,M_v$ and $\delta \in \mathbf{R}^{M_v}$. Define the nonlinear operator G_M:

$$G_M(\mathbf{u}) = \mathcal{R}_M\mathcal{G}(\mathcal{E}_M\mathbf{u}), \qquad \mathbf{u}\in\mathbf{R}^{M_v}$$

This maps $\mathbf{R}^{M_v}$ into $\mathbf{R}^{M_v}$. Recalling (31),

$$L_M(\mathbf{u}) = K_M\mathbf{u} + S_M G_M(\mathbf{u}), \quad \mathbf{u}\in\mathbf{R}^{M_v}$$

Then it is straightforward to obtain

$$\begin{aligned} L'_M(\mathbf{u})\delta &= K_M\delta + S_M G'_M(\mathbf{u})\delta \\ &= K_M\delta + S_M\mathcal{R}_M\mathcal{G}'(\mathcal{E}_M\mathbf{u})\mathcal{E}_M\delta, \qquad \delta\in\mathbf{R}^{M_v} \end{aligned}$$

We need to consider the linear integral equation problem

$$\psi - [\mathcal{K} + \mathcal{S}\mathcal{G}'(u^*)]\psi = f$$

and the collocation method for its solution:

$$\psi_M - \mathcal{P}_M[\mathcal{K} + \mathcal{S}\mathcal{G}'(u^*)]\psi_M = \mathcal{P}_M f$$

with arbitrary $f\in C(\Gamma)$. The associated collocation matrix is $I - A_M$ with

$$A_M = K_M + S_M\mathcal{R}_M\mathcal{G}'(u^*)\mathcal{E}_N$$

By our assumption in Theorems 1 and 2, that 1 is not an eigenvalue of $\mathcal{L}'(u^*)$, it follows that the inverses $[I - \mathcal{P}_M\mathcal{L}'(u^*)]^{-1}$ exist and are uniformly

bounded on $C(\Gamma)$ for all sufficiently large M. From this it follows by standard arguments that the matrices $[I-A_M]^{-1}$ also exist and are uniformly bounded for sufficiently large M, using the matrix row norm. As we asserted earlier in the proofs of Theorems 1 and 2, this leads to the existence and uniform boundedness of $[I- \mathcal{P}_M\mathcal{L}'(u_M^*)]^{-1}$ and $[I - L'_M(\mathbf{u}_M^*)]^{-1}$ with $\mathbf{u}_M^* = \mathcal{R}_M u_M^*$, using (26) and its consequences.

Newton's method for solving $\mathbf{u}_M = L_M(\mathbf{u}_M)$ for $\mathbf{u}_M^*$ is

$$\mathbf{u}_M^{(k+1)} = \mathbf{u}_M^{(k)} - \left[I - L'_M(\mathbf{u}_M^{(k)})\right]^{-1} \left[\mathbf{u}_M^{(k)} - L_M(\mathbf{u}_M^{(k)})\right], \qquad k = 0, 1, \cdots \quad (46)$$

To define our iteration method, we approximate the inverse appearing in this definition. We do so in several steps, to justify the final form of the approximation which we use.

First, consider the approximation

$$\left[I - L'_M(\mathbf{u}_M^{(k)})\right]^{-1} \approx [I - L'_M(\mathbf{u}_M^*)]^{-1}$$

Using this in (46) is known to lead to a linearly convergent iteration method; and this can be shown to be uniformly so for M sufficiently large. As $M \to \infty$, the linear rate of convergence approaches 0.

Next, use

$$[I - L'_M(\mathbf{u}_M^*)]^{-1} \approx [I - L'_M(\mathcal{R}_M u^*)]^{-1}$$

Since $u_M^* \to u^*$, again we have a good approximation of the inverse matrix in (46), provided M is chosen sufficiently large. This new inverse is further approximated by

$$[I - L'_M(\mathcal{R}_M u^*)]^{-1} \approx [I - A_M]^{-1}$$

This is justified by the interpolation convergence result that $\mathcal{E}_M\mathcal{R}_M u^* \to u^*$. Our modification of the Newton iteration (46) now takes the form

$$\mathbf{u}_M^{(k+1)} = \mathbf{u}_M^{(k)} - [I - A_M]^{-1} \left[\mathbf{u}_M^{(k)} - L_M(\mathbf{u}_M^{(k)})\right], \qquad k = 0, 1, \cdots \quad (47)$$

To move between vectors in $\mathbf{R}^{M_v}$ and $\mathbf{R}^{N_v}$, we introduce the restriction operator $\mathcal{R}_{MN}$ and the extension operator $\mathcal{E}_{NM}$. For $\mathbf{u} \in \mathbf{R}^{M_v}$, define $\mathcal{R}_{MN}\mathbf{u} \in \mathbf{R}^{N_v}$ to be the restriction of $\mathbf{u}$ to the nodes associated with the coarse mesh triangulation $\mathcal{T}_N$. For $\mathbf{u} \in \mathbf{R}^{N_v}$, define $\mathcal{E}_{MN}\mathbf{u} \in \mathbf{R}^{M_v}$ to be the

extension of $\mathbf{u}$ to the nodes associated with the fine mesh $\mathcal{T}_M$; and this extension is based on the same interpolation as used in defining $\mathcal{E}_N\mathbf{u}$ in (30).

We now introduce the two-grid approximation in its theoretical framework. Use

$$[I - A_M]^{-1} \approx I + \mathcal{E}_{NM}\,[I - A_N]^{-1}\,\mathcal{R}_{MN}A_M \tag{48}$$

with $N < M$. The triangulation associated with N is usually called the *coarse* mesh, and that associated with M is called the *fine* mesh. This approximation comes out the theory of collectively compact operator approximations. We refer to the extensive discussions in [3, p. 143] for additional discussion and motivation.

To obtain convergence of the iteration with this new approximation, proceed as follows. First, by straightforward algebraic manipulation,

$$\begin{aligned}
&[I - A_M]^{-1} - \left\{I + \mathcal{E}_{NM}\,[I - A_N]^{-1}\,\mathcal{R}_{MN}A_M\right\} \\
&\quad = \quad [I - \mathcal{E}_{NM}(I - A_N)^{-1}\mathcal{R}_{MN}(I - A_M)]\,A_M\,[I - A_M]^{-1} \\
&\quad \equiv \quad C_{NM}\,[I - A_M]^{-1}
\end{aligned} \tag{49}$$

Next, from [7, Appendix], it follows that

$$\|C_{NM}\| \le \frac{5}{3}\left\|[I - \mathcal{P}_N\mathcal{L}'(u^*)]^{-1}\right\|\,\|(\mathcal{P}_N - \mathcal{P}_M)\,\mathcal{L}'(u^*)\| \tag{50}$$

Since $\|(I - \mathcal{P}_M)\,\mathcal{L}'(u^*)\| \to 0$, it follows that

$$\lim_{N\to\infty}\ \sup_{M\ge N}\ \|C_{NM}\| = 0 \tag{51}$$

Finally, we work backward to an approximation based on the iterates actually being used in the computation:

$$[I - A_M]^{-1} \approx I + \mathcal{E}_{NM}\,[I - L'_N(\mathbf{u}_N^*)]^{-1}\,\mathcal{R}_{MN}L'_M(\mathbf{u}_M^{(0)})$$

If N is chosen sufficiently large, and if $\mathbf{u}_M^{(0)}$ is chosen sufficiently close to $\mathbf{u}_M$, then this last approximation will also have small error, uniformly for sufficiently large M.

Combining the above approximations, we define our modified Newton's method:

$$\mathbf{u}_M^{(k+1)} = \mathbf{u}_M^{(k)} - \mathbf{r}_M^{(k)} - \mathcal{E}_{NM}\delta^{(k)} \tag{52}$$

$$[I - L'_N(\mathbf{u}^*_N)]\,\delta^{(k)} = \mathcal{R}_{MN} L'_M(\mathbf{u}^{(0)}_M)\mathbf{r}^{(k)}_M \tag{53}$$

$$\mathbf{r}^{(k)}_M = \mathbf{u}^{(k)}_M - \mathcal{R}_M \mathcal{L}(\mathbf{u}^{(k)}_M) \tag{54}$$

Combining the above arguments, this iteration will converge linearly for all sufficiently large M, uniformly in M, provided the coarse mesh parameter N is chosen sufficiently large, and provided the initial guesses $\mathbf{u}^{(0)}_M$ are chosen sufficiently close to the true solutions $\mathbf{u}^*_M$.

Theorem 3 *Make the same assumptions as in Theorem 1, and consider the iteration (52)-(54). Then there exists an $\epsilon > 0$ and an integer $N_0 \geq 1$ such that if $N \geq N_0$ and if $\mathbf{u}^{(0)}_M$ satisfies*

$$\left\|\mathbf{u}^*_M - \mathbf{u}^{(0)}_M\right\|_\infty \leq \epsilon$$

then

$$\left\|\mathbf{u}^*_M - \mathbf{u}^{(k+1)}_M\right\|_\infty \leq d_{N,M} \left\|\mathbf{u}^*_M - \mathbf{u}^{(k)}_M\right\|_\infty, \quad M \geq N$$

The constant $d_{N,M}$ satisfies

$$\lim_{N\to\infty} \sup_{M>N} d_{N,M} = 0$$

Consequently there is a $\delta \in (0,1)$ for which $d_{N,M} \leq 1 - \delta$, uniformly in M, provided N is chosen sufficiently large.

Proof. The proof consists of simply using the approximations given preceding the theorem and applying standard fixed point arguments. The proof is complicated, but uses standard perturbation arguments. We omit it here. □

5.1 Using the approximate surface

We use the approximate surface $\widetilde{\Gamma}_N$ rather than Γ. We are solving the linear system $\mathbf{u}_M = \widetilde{L}_M(\mathbf{u}_M)$ which has the solution $\widetilde{\mathbf{u}}^*_N$. In analogy with Theorem 3, we have the following.

Theorem 4 *Make the same assumptions as in Theorem 1, and consider the iteration (52)-(54). Let $\widetilde{\Gamma}_N$ be the interpolatory approximating surface defined*

in Section 2. Then there exists an $\epsilon > 0$ and an integer $N_0 \geq 1$ such that if $N \geq N_0$ and if $\tilde{\mathbf{u}}_M^{(0)}$ satisfies

$$\left\| \tilde{\mathbf{u}}_M^* - \tilde{\mathbf{u}}_M^{(0)} \right\|_\infty \leq \epsilon$$

then

$$\left\| \tilde{\mathbf{u}}_M^* - \tilde{\mathbf{u}}_M^{(k+1)} \right\|_\infty \leq \tilde{d}_{N,M} \left\| \tilde{\mathbf{u}}_M^* - \tilde{\mathbf{u}}_M^{(k)} \right\|_\infty, \quad M \geq N$$

The constant $\tilde{d}_{N,M}$ satisfies

$$\lim_{N \to \infty} \sup_{M > N} \tilde{d}_{N,M} = 0$$

Consequently there is a $\delta \in (0,1)$ for which $\tilde{d}_{N,M} \leq 1 - \delta$, uniformly in M, provided N is chosen sufficiently large.

Proof. The proof follows along exactly the same lines as indicated for Theorem 3. The major point at which a problem arises is in showing the analogue of (47)-(51). Let $\tilde{A}_M$ denote the analogue of A_M, when using $\tilde{\Gamma}_N$ rather than Γ.

Using results from [9], we can show

$$\left\| A_M - \tilde{A}_M \right\| = O(\hat{h}^2), \quad M \geq 1, \quad \hat{h}^2 M = O(1) \tag{55}$$

A proof can be given which is almost exactly the same as that of [9, Theorem 5.2]. Combining this with (47)-(51), we can give a proof that

$$\left[I - \tilde{A}_M \right]^{-1} \approx I + \mathcal{E}_{NM} \left[I - \tilde{A}_N \right]^{-1} \mathcal{R}_{MN} \tilde{A}_M, \qquad M \geq N \tag{56}$$

when N is chosen sufficiently large. □

6 ITERATION EXAMPLES

We use the same surfaces and examples as were used in Section 4, but now we consider the solution of the nonlinear systems $\mathbf{u}_M = \tilde{L}_M(\mathbf{u}_M)$ associated with those earlier examples. We use the iteration (52)-(54), modified to use the approximate surface $\tilde{\Gamma}_M$. In our examples, the initial guess $\mathbf{u}_M^{(0)}$ is obtained from the final solution for the preceding value of M. Where possible, we

Table 3: Iteration Statistics for $S\#1$.

N	M	*Ratio*	*Cost*
8	32	.584	.002
	128	.545	.29
	512	.550	5.0
32	128	.279	.344
	512	.288	5.8
20	80	.301	.133
	320	.362	1.90
80	320	.210	2.25

Table 4: Iteration Statistics for $S\#2$.

N	M	*Ratio*	*Cost*
8	32	*diverges*	
32	128	.289	.340
	512	.320	5.73
20	80	.381	.117
	320	.405	1.87
80	320	.192	2.27

experimented with different values of N, to see the effect of this on the speed of convergence.

In Tables 3 and 4, we give results of the two-grid iteration for the surfaces $S\#1$ and $S\#2$. We iterated until the successive differences $\left\|\widetilde{\mathbf{u}}_M^{(k)} - \widetilde{\mathbf{u}}_M^{(k-1)}\right\|_\infty$ were less than 10^{-10}. The unknown function used was $u_1(x,y,z) = x^2 + y^2 - 2z^2$. For $S\#1$, the results for u_1 and u_2 (also given in Section 4) were almost identical. For $S\#2$, the results when using u_2 were slightly slower; but the essential nature of the results was the same, and therefore we omit them here. The column labeled *Ratio* is either the final value of the ratio

$$\frac{\left\|\widetilde{\mathbf{u}}_M^{(k)} - \widetilde{\mathbf{u}}_M^{(k-1)}\right\|_\infty}{\left\|\widetilde{\mathbf{u}}_M^{(k-1)} - \widetilde{\mathbf{u}}_M^{(k-2)}\right\|_\infty},$$

or for those cases where this ratio was still varying, *Ratio* is the geometric average of the last several values of this ratio. The column labeled *Cost* is the approximate cost in time (in seconds) of calculating each iterate $\widetilde{\mathbf{u}}_M^{(k)}$. The results were computed on an Hewlett-Packard 720, which operates at 57 MIPS. It can be seen that the speed of convergence improves as the coarse mesh parameter N is increased.

6.1 Comparative timings

It is useful to look at timings of all parts of the boundary integral equation method which is described in this paper, to see which parts are the more expensive. To this end, we give such timings for the problem on $S\#1$ with u_1 as the solution being sought. To look at the entire picture, we include the parameter N_d [see Section 4 following (43)] and the errors $\|\mathcal{R}_M u^* - \widetilde{\mathbf{u}}_M^*\|_\infty$. The timings are affected greatly by N_d. As it increases, the time to set up the collocation matrices K_M and S_M also increases greatly; and if N_d is chosen too small, then $error \equiv \|\mathcal{R}_M u^* - \widetilde{\mathbf{u}}_M^*\|_\infty$ is not as small as it might be otherwise.

The timings (in seconds) are given in Table 5. The columns labeled *triang, matrix,* and *iterate* refer, respectively, to the costs of the triangulation, the setup of the collocation matrix, and the calculation of one iterate. The timings are somewhat crude, as they are taken on a system with multi-tasking (although the timings were done when there were no other users on

Table 5: Timings (in seconds) on $S\#1$

N	M	N_d	*error*	*triang*	*matrix*	*iterate*
8	32	0	5.77E−2	.002	3.86	.02
		1	5.72E−2	.002	6.25	.02
8	128	0	5.47E−3	.11	32.1	.26
		1	4.48E−3	.10	47.9	.29
		2	4.48E−3	.10	128	.28
8	512	0	1.37E−3	1.21	414	4.77
		1	2.87E−4	1.23	487	4.98
		2	2.88E−4	1.22	813	4.98

the machine). Nonetheless, the timings serve to show the relative costs of the various parts of the program; and they show the effect of increasing N_d.

The setup of the collocation matrices K_M and S_M is the most costly part of the solution process. This matrix setup cost is almost entirely due to the numerical integration of the collocation integrals, and this was discussed earlier in the beginning of Section 4. We have experimented a great deal with the numerical integration, and the present approach is the most efficient of those we have implemented. From the timings in the table, there seems little reason to improve further the efficiency of the iteration method, at least until the matrix setup cost can be reduced significantly.

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