TWO-POINT BVP

Consider the two-point boundary value problem of a second-order linear equation:

$$Y''(x) = p(x)\,Y'(x) + q(x)\,Y(x) + r(x)$$
$$a \le x \le b$$
$$Y(a) = g_1, \quad Y(b) = g_2$$

Assume the given functions p, q and r are continuous on $[a, b]$. Unlike the initial value problem of the equation that always has a unique solution, the theory of the two-point boundary value problem is more complicated. We will assume the problem has a unique smooth solution Y; a sufficient condition for this is $q(x) > 0$ for $x \in [a, b]$.

In general, we need to depend on numerical methods to solve the problem.

FINITE DIFFERENCE METHOD

We derive a finite difference scheme for the two-point boundary value problem in three steps.

Step 1. Discretize the interval $[a, b]$.
Let N be a positive integer, and divide the interval $[a, b]$ into N equal parts:

$$[a, b] = [x_0, x_1] \cup [x_1, x_2] \cup \cdots \cup [x_{N-1}, x_N]$$

Let $h = (b - a)/N$, called stepsize or gridsize. $x_i = a + i\,h$, $0 \le i \le N$, are grid (or node) points.

We use the notation $p_i = p(x_i)$, $q_i = q(x_i)$, $r_i = r(x_i)$, $0 \le i \le N$. For $0 \le i \le N$, y_i is numerical approximation of $Y_i = Y(x_i)$.

Step 2. Discretize the differential equation at the interior node points $x_1, \ldots, x_{N-1}$.
Recall

$$Y'(x_i) = \frac{Y_{i+1} - Y_{i-1}}{2\,h} + O(h^2)$$
$$Y''(x_i) = \frac{Y_{i+1} - 2\,Y_i + Y_{i-1}}{h^2} + O(h^2)$$

Then the differential equation at $x = x_i$ becomes

$$\frac{Y_{i+1} - 2\,Y_i + Y_{i-1}}{h^2} = p_i \frac{Y_{i+1} - Y_{i-1}}{2\,h} + q_i Y_i + r_i + O(h^2)$$

Dropping the $O(h^2)$ term, replacing Y_i by y_i, we obtain the difference equations

$$\frac{y_{i+1} - 2\,y_i + y_{i-1}}{h^2} = p_i \frac{y_{i+1} - y_{i-1}}{2\,h} + q_i y_i + r_i$$

for $1 \le i \le N-1$. So

$$-\left(1 + \frac{h}{2} p_i\right) y_{i-1} + (2 + h^2 q_i) y_i + \left(\frac{h}{2}\,p_i - 1\right) y_{i+1} = -h^2 r_i, \quad 1 \le i \le N-1$$

Step 3. Treatment of boundary conditions.
Use

$$y_0 = g_1, \quad y_N = g_2$$

Then the difference equation with $i = 1$ becomes

$$\begin{aligned}(2 + h^2 q_1)\, y_1 + \left(\frac{h}{2} p_1 - 1\right) y_2 \\ = -h^2 r_1 + \left(1 + \frac{h}{2}\, p_1\right) g_1\end{aligned}$$

and that with $i = N - 1$

$$\begin{aligned}-\left(1 + \frac{h}{2}\, p_{N-1}\right) y_{N-2} + (2 + h^2 q_{N-1})\, y_{N-1} \\ = -h^2 r_{N-1} + \left(1 - \frac{h}{2} p_{N-1}\right) g_2\end{aligned}$$

Finally, the finite difference system is

$$A\mathbf{y} = \mathbf{b}$$

where, unknown numerical solution vector

$$\mathbf{y} = [y_1, \cdots, y_{N-1}]^T$$

right-hand side vector

$$\mathbf{b} = \left[-h^2 r_1 + \left(1 + \frac{h}{2} p_1\right) g_1, -h^2 r_2, \cdots, \right.$$
$$\left. -h^2 r_{N-2}, -h^2 r_{N-1} + \left(1 - \frac{h}{2} p_{N-1}\right) g_2\right]^T$$

and coefficient matrix A, which is tridiagonal.

THEORETICAL RESULTS

Suppose the true solution $Y(x)$ has several continuous derivatives. For the finite difference scheme, we have the following results.

1. The scheme is of second-order accurate,

$$\max_{0 \le i \le N} |Y(x_i) - y_i| = O(h^2)$$

2. There is an asymptotic error expansion

$$Y(x_i) - y_h(x_i) = h^2 D(x_i) + O(h^4)$$

   for some function $D(x)$ independent of h.

Define Richardson extrapolation

$$\tilde{y}_h(x_i) = \frac{4\, y_h(x_i) - y_{2h}(x_i)}{3}$$

Then

$$Y(x_i) - \tilde{y}_h(x_i) = O(h^4)$$

i.e., without much additional effort, we obtain a fourth-order approximate solution.

Actually we can have more terms in asymptotic error expansion

$$Y(x_i) - y_h(x_i) = h^2 D_1(x_i) + h^4 D_2(x_i) + O(h^6)$$

for some functions $D_1(x)$ and $D_2(x)$ independent of h.

We can then perform further steps of extrapolation

$$Y(x_i) - \frac{16\,\tilde{y}_h(x_i) - \tilde{y}_{2h}(x_i)}{15} = O(h^6)$$

to get even higher order convergence.

EXAMPLE. Use the finite difference method to solve the boundary value problem

$$\begin{aligned} Y'' &= -\frac{2\,x}{1+x^2}Y' + Y + \frac{2}{1+x^2} \\ &\qquad - \log(1+x^2), \quad 0 \le x \le 1 \\ Y(0) &= 0 \\ Y(1) &= \log(2) \end{aligned}$$

The true solution is $Y(x) = \log(1+x^2)$.

Numerical errors $Y(x) - y_h(x)$

x	$h = 1/20$	$h = 1/40$	R	$h = 1/80$	R
0.1	5.10E − 5	1.27E − 5	4.0	3.18E − 6	4.0
0.2	7.84E − 5	1.96E − 5	4.0	4.90E − 6	4.0
0.3	8.64E − 5	2.16E − 5	4.0	5.40E − 6	4.0
0.4	8.08E − 5	2.02E − 5	4.0	5.05E − 6	4.0
0.5	6.73E − 5	1.68E − 5	4.0	4.21E − 6	4.0
0.6	5.08E − 5	1.27E − 5	4.0	3.17E − 6	4.0
0.7	3.44E − 5	8.60E − 6	4.0	2.15E − 6	4.0
0.8	2.00E − 5	5.01E − 6	4.0	1.25E − 6	4.0
0.9	8.50E − 6	2.13E − 6	4.0	5.32E − 7	4.0

The column marked "R" next to the column of the solution errors for a stepsize h consists of the ratios of the solution errors for the stepsize h with those for the stepsize $2h$. We clearly observe an error reduction of a factor of around 4 when the stepsize is halved, indicating a second order convergence of the method.

The next table give the extrapolation errors for solving the boundary value problem, showing the accuracy improvement by the extrapolation.

x	$h = 1/40$	$h = 1/80$	R
0.1	$-9.23\text{E}-09$	$-5.76\text{E}-10$	16.01
0.2	$-1.04\text{E}-08$	$-6.53\text{E}-10$	15.99
0.3	$-6.60\text{E}-09$	$-4.14\text{E}-10$	15.96
0.4	$-1.18\text{E}-09$	$-7.57\text{E}-11$	15.64
0.5	$3.31\text{E}-09$	$2.05\text{E}-10$	16.14
0.6	$5.76\text{E}-09$	$3.59\text{E}-10$	16.07
0.7	$6.12\text{E}-09$	$3.81\text{E}-10$	16.04
0.8	$4.88\text{E}-09$	$3.04\text{E}-10$	16.03
0.9	$2.67\text{E}-09$	$1.67\text{E}-10$	16.02

DIFFERENCE SCHEME FOR GENERAL EQUATION

Difference schemes for solving boundary value problems of more general equations can be derived similarly. As an example, consider

$$Y'' = f(x, Y, Y')$$

At an interior node point x_i, the differential equation can be approximated by the difference equation

$$\frac{y_{i+1} - 2\,y_i + y_{i-1}}{h^2} = f\left(x_i, y_i, \frac{y_{i+1} - y_{i-1}}{2\,h}\right)$$

TREATMENT OF OTHER BOUNDARY CONDITIONS

Boundary conditions involving the derivative of the unknown need to be discretized carefully.

Consider the following boundary condition at $x = b$:

$$Y'(b) + k\,Y(b) = g_2$$

If we use the discrete boundary condition

$$\frac{y_N - y_{N-1}}{h} + k\,y_N = g_2$$

then the difference solution will have a first-order accuracy only, even though the difference equations at the interior nodes are second-order.

To maintain second-order accuracy, need a second-order treatment of the derivative term $Y'(b)$, e.g., since

$$Y'(b) = \frac{3\,Y_N - 4\,Y_{N-1} + Y_{N-2}}{2\,h} + O(h^2)$$

we can approximate the boundary condition by

$$\frac{3\,y_N - 4\,y_{N-1} + y_{N-2}}{2\,h} + k\,y_N = g_2$$