Consider the pendulum shown below. Assume the rod is of negligible mass, that the pendulum is of mass $m$, and that the rod is of length $\ell$. Assume the pendulum moves in the plane shown, and assume there is no friction in the motion about its pivot point. Let $\theta(x)$ denote the position of the pendulum about the vertical line thru the pivot, with $\theta$ measured in radians and $x$ measured in units of time. Then Newton’s second law implies

$$m\ell \frac{d^2 \theta}{dx^2} = -mg \sin (\theta (x))$$
Introduce $Y_1(x) = \theta(x)$ and $Y_2(x) = \theta'(x)$. The function $Y_2(x)$ is called the angular velocity. We can now write

$$Y_1'(x) = Y_2(x), \quad Y_1(0) = \theta(0)$$
$$Y_2'(x) = -\frac{g}{\ell} \sin(Y_1(x)), \quad Y_2(0) = \theta'(0)$$

This is a simultaneous system of two differential equations in two unknowns.

We often write this in vector form. Introduce

$$\mathbf{Y}(x) = \begin{bmatrix} Y_1(x) \\ Y_2(x) \end{bmatrix}$$

Then

$$\mathbf{Y}'(x) = \begin{bmatrix} Y_2(x) \\ -\frac{g}{\ell} \sin(Y_1(x)) \end{bmatrix}$$
$$\mathbf{Y}(0) = \mathbf{Y}_0 = \begin{bmatrix} Y_1(0) \\ Y_2(0) \end{bmatrix} = \begin{bmatrix} \theta(0) \\ \theta'(0) \end{bmatrix}$$
Introduce
\[
f(x, z) = \begin{bmatrix} z_2 \\ -\frac{g}{\ell} \sin(z_1) \end{bmatrix}, \quad z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}
\]

Then our differential equation problem
\[
Y'(x) = \begin{bmatrix} Y_2(x) \\ -\frac{g}{\ell} \sin(Y_1(x)) \end{bmatrix}
\]
\[
Y(0) = Y_0 = \begin{bmatrix} Y_1(0) \\ Y_2(0) \end{bmatrix} = \begin{bmatrix} \theta(0) \\ \theta'(0) \end{bmatrix}
\]
can be written in the familiar form
\[
Y'(x) = f(x, Y(x)), \quad Y(0) = Y_0 \quad (1)
\]
We can convert any higher order differential equation into a system of first order differential equations, and we can write them in the vector form (1).
Lotka-Volterra predator-prey model.

\[
\begin{align*}
Y_1' &= AY_1[1 - BY_2], \quad Y_1(0) = Y_{1,0} \\
Y_2' &= CY_2[D Y_1 - 1], \quad Y_2(0) = Y_{2,0}
\end{align*}
\]

with \(A, B, C, D > 0\). \(x\) denotes time, \(Y_1(x)\) is the number of prey (e.g., rabbits) at time \(x\), and \(Y_2(x)\) the number of predators (e.g., foxes). If there is only a single type of predator and a single type of prey, then this model is often a good approximation of reality.

Again write

\[
\mathbf{Y}(x) = \begin{bmatrix} Y_1(x) \\ Y_2(x) \end{bmatrix}
\]

and define

\[
f(x, z) = \begin{bmatrix} Az_1[1 - Bz_2] \\ Cz_2[Dz_1 - 1] \end{bmatrix}, \quad z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}
\]

although there is no explicit dependence on \(x\). Then system (2) can be written as

\[
\mathbf{Y}'(x) = f(x, \mathbf{Y}(x)), \quad \mathbf{Y}(0) = \mathbf{Y}_0
\]
GENERAL SYSTEMS OF ODES

An initial value problem for a system of $m$ differential equations has the form

$$
Y_1'(x) = f_1(x, Y_1(x), \ldots, Y_m(x)), \quad Y_1(x_0) = Y_{1,0} \\
\vdots \\
Y_m'(x) = f_m(x, Y_1(x), \ldots, Y_m(x)), \quad Y_m(x_0) = Y_{m,0}
$$

(3)

Introduce

$$
Y(x) = \begin{bmatrix}
Y_1(x) \\
\vdots \\
Y_m(x)
\end{bmatrix}, \quad Y_0 = \begin{bmatrix}
Y_{1,0} \\
\vdots \\
Y_{m,0}
\end{bmatrix}
$$

$$
f(x, z) = \begin{bmatrix}
f_1(x, z_1, \ldots, z_m) \\
\vdots \\
f_m(x, z_1, \ldots, z_m)
\end{bmatrix}
$$

Then (3) can be written as

$$
Y'(x) = f(x, Y(x)), \quad Y(0) = Y_0
$$
LINEAR SYSTEMS

Of special interest are systems of the form

\[ Y'(x) = AY(x) + G(x), \quad Y(0) = Y_0 \]  \hspace{1cm} (4)

with \( A \) a square matrix of order \( m \) and \( G(x) \) a column vector of length \( m \) with functions \( G_i(x) \) as components. Using the notation introduced for writing systems,

\[ f(x, z) = Az + G(x), \quad z \in \mathbb{R}^m \]

This equation is the analogue for studying systems of ODEs that the model equation

\[ y' = \lambda y + g(x) \]

is for studying a single differential equation.
Consider
\[ Y'(x) = f(x, Y(x)), \quad Y(0) = Y_0 \]
to be a systems of two equations
\[ \begin{align*}
Y'_1(x) &= f_1(x, Y_1(x), Y_2(x)), \quad Y_1(0) = Y_{1,0} \\
Y'_2(x) &= f_2(x, Y_1(x), Y_2(x)), \quad Y_2(0) = Y_{2,0}
\end{align*} \tag{5} \]
Denote its solution be \([Y_1(x), Y_2(x)]\).
Following the earlier derivations for Euler’s method, we can use Taylor’s theorem to obtain
\[ \begin{align*}
Y_1(x_{n+1}) &= Y_1(x_n) + hf_1(x_n, Y_1(x_n), Y_2(x_n)) + \frac{h^2}{2} Y_1''(\xi_n) \\
Y_2(x_{n+1}) &= Y_2(x_n) + hf_2(x_n, Y_1(x_n), Y_2(x_n)) + \frac{h^2}{2} Y_2''(\xi_n)
\end{align*} \tag{6} \]
Dropping the remainder terms, we obtain Euler’s method for problem (5),
\[ \begin{align*}
y_{1,n+1} &= y_{1,n} + hf_1(x_n, y_{1,n}, y_{2,n}), \quad y_{1,0} = Y_{1,0} \\
y_{2,n+1} &= y_{2,n} + hf_2(x_n, y_{1,n}, y_{2,n}), \quad y_{2,0} = Y_{2,0}
\end{align*} \]
for \( n = 0, 1, 2, \ldots \).
ERROR ANALYSIS

If $Y_1(x)$, $Y_2(x)$ are twice continuously differentiable, and if the functions $f_1(x, z_1, z_2)$ and $f_2(x, z_1, z_2)$ are sufficiently differentiable, then it can be shown that

$$
\max_{x_0 \leq x \leq b} \left| Y_1(x_n) - y_{1,n} \right| \leq ch
$$

$$
\max_{x_0 \leq x \leq b} \left| Y_2(x_n) - y_{2,n} \right| \leq ch
$$

(7)

for a suitable choice of $c \geq 0$.

The theory depends on generalizations of the proof used with Euler’s method for a single equation. One needs to assume that there is a constant $K > 0$ such that

$$
\| f(x, z) - f(x, w) \|_\infty \leq K \| z - w \|_\infty
$$

(8)

for $x_0 \leq x \leq b$, $z, w \in \mathbb{R}^2$. Recall the definition of the norm $\| \cdot \|_\infty$ from Chapter 6.
The role of $\frac{\partial f(x, z)}{\partial z}$ in the single variable theory is replaced by the Jacobian matrix

$$
F(x, z) = \begin{bmatrix}
\frac{\partial f_1(x, z_1, z_2)}{\partial z_1} & \frac{\partial f_1(x, z_1, z_2)}{\partial z_2} \\
\frac{\partial f_2(x, z_1, z_2)}{\partial z_1} & \frac{\partial f_2(x, z_1, z_2)}{\partial z_2}
\end{bmatrix}
$$

(9)

It is possible to show that

$$
K = \max_{x_0 \leq x \leq b, \mathbf{z} \in \mathbb{R}^2} \|F(x, \mathbf{z})\|_\infty
$$

is suitable for showing (8).

All of this work generalizes to problems of any order $m \geq 2$. Then we require

$$
\|f(x, \mathbf{z}) - f(x, \mathbf{w})\|_\infty \leq K \|\mathbf{z} - \mathbf{w}\|_\infty
$$

(10)

with $x_0 \leq x \leq b$, $\mathbf{z}, \mathbf{w} \in \mathbb{R}^m$. The choice of $K$ is often obtained using

$$
K = \max_{x_0 \leq x \leq b, \mathbf{z} \in \mathbb{R}^m} \|F(x, \mathbf{z})\|_\infty
$$

where $F(x, \mathbf{z})$ is the $m \times m$ generalization of (9).
The Euler method in all cases can be written in the dimensionless form

\[ y_{n+1} = y_n + hf(x_n, y_n), \quad n \geq 0 \]

with \( y_0 = Y_0 \).

It can be shown that if (10) is satisfied, and if \( Y(x) \) is twice-continuously differentiable on \([x_0, b]\), then

\[
\max_{x_0 \leq x \leq b} \| Y(x_n) - y_n \|_{\infty} \leq ch
\]

for some \( c \geq 0 \) and for all small values of \( h \).

In addition, we can show there is a vector function \( D(x) \) for which

\[
Y(x) - y_h(x) = D(x)h + O(h^2), \quad x_0 \leq x_n \leq b
\]

for \( x = x_0, x_1, \ldots, b \). Here \( y_h(x) \) shows the dependence of the solution on \( h \), and \( y_h(x) = y_n \) for \( x = x_0 + nh \). This justifies the use of Richardson extrapolation, leading to

\[
Y(x) - y_h(x) = y_h(x) - y_{2h}(x) + O(h^2)
\]
NUMERICAL EXAMPLE. Consider solving the initial value problem

\[ Y''' + 3Y'' + 3Y' + Y = -4 \sin(x), \]
\[ Y(0) = Y'(0) = 1, \quad Y''(0) = -1 \]  \hspace{1cm} (12)

Reformulate it as

\[ Y_1' = Y_2 \]
\[ Y_2' = Y_3 \]
\[ Y_3' = -Y_1 - 3Y_2 - 3Y_3 - 4 \sin(x), \quad Y_3(0) = -1 \]  \hspace{1cm} (13)

The solution of (12) is \( Y(x) = \cos(x) + \sin(x) \), and the solution of (13) can be generated from it using \( Y_1(x) = Y(x) \).
The results for $Y_1(x) = \sin(x) + \cos(x)$ are given in the following table, for stepsizes $2h = 0.1$ and $h = 0.05$.

The Richardson error estimate is quite accurate.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y(x)$</th>
<th>$y(x) - y_{2h}(x)$</th>
<th>$y(x) - y_h(x)$</th>
<th>Ratio</th>
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<tr>
<td>2</td>
<td>0.49315</td>
<td>-8.78E - 2</td>
<td>-4.25E - 2</td>
<td>2.1</td>
</tr>
<tr>
<td>4</td>
<td>-1.41045</td>
<td>1.39E - 1</td>
<td>6.86E - 2</td>
<td>2.0</td>
</tr>
<tr>
<td>6</td>
<td>0.68075</td>
<td>5.19E - 2</td>
<td>2.49E - 2</td>
<td>2.1</td>
</tr>
<tr>
<td>8</td>
<td>0.84386</td>
<td>-1.56E - 1</td>
<td>-7.56E - 2</td>
<td>2.1</td>
</tr>
<tr>
<td>10</td>
<td>-1.38309</td>
<td>8.39E - 2</td>
<td>4.14E - 2</td>
<td>2.0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y(x)$</th>
<th>$y(x) - y_h(x)$</th>
<th>$y_h(x) - y_{2h}(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.49315</td>
<td>-4.25E - 2</td>
<td>-4.53E - 2</td>
</tr>
<tr>
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<td>-1.41045</td>
<td>6.86E - 2</td>
<td>7.05E - 2</td>
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<td>2.49E - 2</td>
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<td>8</td>
<td>0.84386</td>
<td>-7.56E - 2</td>
<td>-7.99E - 2</td>
</tr>
<tr>
<td>10</td>
<td>-1.38309</td>
<td>4.14E - 2</td>
<td>4.25E - 2</td>
</tr>
</tbody>
</table>
OTHER METHODS

Other numerical methods apply to systems in the same straightforward manner. By using the vector form

\[ Y'(x) = f(x, Y(x)), \quad Y(0) = Y_0 \]  

(14)

for a system, there is no apparent change in the numerical method. For example, the following Runge-Kutta method for solving a single differential equation,

\[ y_{n+1} = y_n + \frac{h}{2} [f(x_n, y_n) + f(x_{n+1}, y_n + hf(x_n, y_n))], \quad n \geq 0, \]

generalizes as follows for solving (14):

\[ y_{n+1} = y_n + \frac{h}{2} [f(x_n, y_n) + f(x_{n+1}, y_n + hf(x_n, y_n))], \quad n \geq 0. \]

This can then be decomposed into components if needed. For a system of order 2, we have

\[ y_{j,n+1} = y_{j,n} + \frac{h}{2} \left[ f_j(x_n, y_{1,n}, y_{2,n}) + f_j(x_{n+1}, y_{1,n} + hf_1(x_n, y_{1,n}, y_{2,n}), \right. \]
\[ \left. y_{2,n} + hf_2(x_n,y_{1,n}, y_{2,n}) \right] \]

for \( n \geq 0 \) and \( j = 1, 2. \)