BINARY INTEGERS

A binary integer $x$ is a finite sequence of the digits 0 and 1, which we write symbolically as

$$x = (a_m a_{m-1} \cdots a_2 a_1 a_0)_2$$

where I insert the parentheses with subscript $(\cdot)_2$ in order to make clear that the number is binary. The above has the decimal equivalent

$$x = a_m 2^m + a_{m-1} 2^{m-1} + \cdots + a_1 2^1 + a_0$$

For example, the binary integer $x = (110101)_2$ has the decimal value

$$x = 2^5 + 2^4 + 2^2 + 2^0 = 53$$

The binary integer $x = (111 \cdots 1)_2$ with $m$ ones has the decimal value

$$x = 2^{m-1} + \cdots + 2^1 + 1 = 2^m - 1$$
DECIMAL TO BINARY INTEGER CONVERSION

Given a decimal integer $x$ we write

\[ x = (a_m a_{m-1} \cdots a_2 a_1 a_0)_2 = a_m 2^m + a_{m-1} 2^{m-1} + \cdots + a_1 2^1 + a_0 \]

Divide $x$ by 2, calling the quotient $x_1$. The remainder is $a_0$, and

\[ x_1 = a_m 2^{m-1} + a_{m-1} 2^{m-2} + \cdots + a_1 2^0 \]

Continue the process. Divide $x_1$ by 2, calling the quotient $x_2$. The remainder is $a_1$, and

\[ x_2 = a_m 2^{m-2} + a_{m-1} 2^{m-3} + \cdots + a_2 2^0 \]

After a finite number of such steps, we will obtain all of the coefficients $a_i$, and the final quotient will be zero.

Try this with a few decimal integers.
EXAMPLE

The following shortened form of the above method is convenient for hand computation. Convert \((11)_10\) to binary.

\[
\begin{align*}
\lfloor 2\sqrt{11} \rfloor &= 5 &= x_1 & a_0 = 1 \\
\lfloor 2\sqrt{5} \rfloor &= 2 &= x_2 & a_1 = 1 \\
\lfloor 2\sqrt{2} \rfloor &= 1 &= x_3 & a_2 = 0 \\
\lfloor 2\sqrt{1} \rfloor &= 0 &= x_4 & a_3 = 1
\end{align*}
\]

In this, the notation \(\lfloor b \rfloor\) denotes the largest integer \(\leq b\), and the notation \(2\sqrt{n}\) denotes the quotient resulting from dividing 2 into \(n\). From the above calculation, \((11)_10 = (1011)_2\).
A binary fraction $x$ is a sequence (possibly infinite) of the digits 0 and 1:

$$x = (a_1a_2a_3 \cdots a_m \cdots)_2$$

$$= a_12^{-1} + a_22^{-2} + a_32^{-3} + \cdots$$

For example, $x = ( .1101)_2$ has the decimal value

$$x = 2^{-1} + 2^{-2} + 2^{-4}$$

$$= .5 + .25 + .0625 = 0.8125$$

BINARY FRACTIONS
Recall the formula for the geometric series

\[ \sum_{i=0}^{n} r^i = \frac{1 - r^{n+1}}{1 - r}, \quad r \neq 1 \]

Letting \( n \to \infty \) with \( |r| < 1 \), we obtain the formula

\[ \sum_{i=0}^{\infty} r^i = \frac{1}{1 - r}, \quad |r| < 1 \]

Using this,

\[ (.010101010101010\cdots)_2 = 2^{-2} + 2^{-4} + 2^{-6} + \cdots \]

\[ = 2^{-2} (1 + 2^{-2} + 2^{-4} + \cdots) \]

which sums to the fraction \( 1/3 \).

Also,

\[ (.11001100110011\cdots)_2 \]

\[ = 2^{-1} + 2^{-2} + 2^{-5} + 2^{-6} + \cdots \]

and this sums to the decimal fraction \( 0.8 = \frac{8}{10} \).
DECIMAL TO BINARY FRACTION CONVERSION

In
\[ x_1 = (a_1a_2a_3\cdots a_m\cdots)_2 = a_12^{-1} + a_22^{-2} + a_32^{-3} + \cdots \]
we multiply by 2. The integer part will be \(a_1\); and after it is removed we have the binary fraction
\[ x_2 = (a_2a_3\cdots a_m\cdots)_2 = a_22^{-1} + a_32^{-2} + a_42^{-3} + \cdots \]
Again multiply by 2, obtaining \(a_2\) as the integer part of \(2x_2\). After removing \(a_2\), let \(x_3\) denote the remaining number. Continue this process as far as needed.

For example, with \(x = \frac{1}{5}\), we have
\[ x_1 = .2; \quad 2x_1 = .4; \quad x_2 = .4 \text{ and } a_1 = 0 \]
\[ 2x_2 = .8; \quad x_3 = .8 \text{ and } a_2 = 0 \]
\[ 2x_3 = 1.6; \quad x_4 = .6 \text{ and } a_2 = 1 \]
Continue this to get the pattern
\[ (.2)_{10} = (.00110011001100\cdots)_2 \]
## ADDITION TABLE

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