A Spectral Method for the Biharmonic Equation

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Abstract Let Ω be an open, simply connected, and bounded region in $\mathbb{R}^d$, $d \geq 2$, with a smooth boundary $\partial\Omega$ that is homeomorphic to $\mathbb{S}^{d-1}$. Consider solving $\Delta^2 u + \gamma u = f$ over Ω with zero Dirichlet boundary conditions. A Galerkin method based on a polynomial approximation space is proposed, yielding an approximation u_n. With sufficiently smooth problem parameters, the method is shown to be rapidly convergent. For $u \in C^\infty\left(\overline{\Omega}\right)$ and assuming $\partial\Omega$ is a C^∞ boundary, the convergence of $\|u - u_n\|_{H^2(\Omega)}$ to zero is faster than any power of $1/n$. Numerical examples illustrate experimentally an exponential rate of convergence.

1 Introduction

Consider the biharmonic problem

$$\Delta^2 u(s) + \gamma(s)\, u(s) = f(s), \qquad s \in \Omega, \tag{1}$$

with the Dirichlet boundary conditions

$$u(s) = \frac{\partial u(s)}{\partial n_s} = 0, \qquad s \in \partial\Omega. \tag{2}$$

The region $\Omega \subseteq \mathbb{R}^d$, $d \geq 2$, is to be bounded and simply-connected; and its boundary $\partial\Omega$ is to be smooth and homeomorphic to $\mathbb{S}^{d-1}$. Assume $f \in L^2(\Omega)$ and

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$$\gamma_{\min} \equiv \min_{s\in\overline{\Omega}} \gamma(s) \geq 0. \tag{3}$$

This can be looked upon as a problem in the Sobolev space $H^4(\Omega)$. It can also be reformulated as a variational problem. For background on the use of this problem in mechanics, see [12], [20, Chap. 8].

Introduce the bilinear functional

$$\mathscr{A}(u,v) = \int_\Omega [\Delta u(s)\,\Delta v(s) + \gamma(s)\,u(s)\,v(s)]\,ds$$

and the linear functional

$$\ell_f(v) \equiv (f,v) = \int_\Omega f(s)\,v(s)\,ds, \qquad v \in L^2(\Omega).$$

Introduce the Hilbert space

$$H_0^2(\Omega) = \left\{ v \in H^2(\Omega) \mid v, \frac{\partial v}{\partial n} = 0, \text{ on } \partial\Omega \right\}.$$

For the norm, use

$$\|v\|_2 \equiv \|v\|_{H^2(\Omega)} = \sqrt{\sum_{|\mathbf{k}|\leq 2} \|D^{\mathbf{k}}v\|^2_{L^2(\Omega)}},$$

where $\mathbf{k} = (k_1,\ldots,k_d)$, $|\mathbf{k}| = k_1 + \cdots + k_d$, and

$$D^{\mathbf{k}}v(s) = \frac{\partial^{|\mathbf{k}|} v(s_1,\ldots,s_d)}{\partial s_1^{k_1}\cdots\partial s_d^{k_d}}.$$

The variational formulation of (1)-(2) is to find $u \in H_0^2(\Omega)$ for which

$$\mathscr{A}(u,v) = \ell_f(v), \qquad \forall v \in H_0^2(\Omega). \tag{4}$$

For a discussion of this reformulation, see Ciarlet [10, p. 28]. With the above assumptions and definitions, $\mathscr{A}$ is a strongly elliptic operator on $H_0^2(\Omega)$,

$$\mathscr{A}(v,v) \geq c_e \|v\|_2^2, \qquad v \in H_0^2(\Omega),$$

with $c_e > 0$. Also, $\mathscr{A}$ is a bounded bilinear operator,

$$|\mathscr{A}(v,w)| \leq c_{\mathscr{A}} \|v\|_2 \|w\|_2, \qquad v,w \in H_0^2(\Omega),$$

for some finite $c_{\mathscr{A}} > 0$. Finally,

$$\|\ell_f\| \leq \|f\|_{L^2(\Omega)}.$$

The Lax-Milgram Theorem (cf. [7, §8.3], [9, §2.7]) implies the existence of a unique solution u to (4) with

$$\|u\|_2 \le \frac{1}{c_e}\|\ell_f\|. \tag{5}$$

In §3 we present a Galerkin method for approximating (4), making use of multivariate orthonormal polynomial approximations. Numerical examples are given in §4.

2 Preliminaries

Assume the existence of an explicitly known continuously differentiable mapping

$$\Phi : \overline{\mathbb{B}}^d \xrightarrow[onto]{1-1} \overline{\Omega} \tag{6}$$

and let $\Psi = \Phi^{-1} : \overline{\Omega} \xrightarrow[onto]{1-1} \overline{\mathbb{B}}^d$ denote the inverse mapping. A very simple example of such a mapping is when Ω is the ellipse

$$\left(\frac{s_1}{a}\right)^2 + \left(\frac{s_2}{b}\right)^2 \le 1$$

with $a, b > 0$. Choose

$$\Phi(x) = (ax_1, bx_2), \qquad x \in \mathbb{B}^2.$$

It is necessary to know Φ explicitly, but not Ψ.The creation of such a mapping Φ is examined at length in [5].

Let

$$J(x) \equiv (D\Phi)(x) = \begin{bmatrix} \frac{\partial \Phi_1(x)}{\partial x_1} & \cdots & \frac{\partial \Phi_1(x)}{\partial x_d} \\ \vdots & \ddots & \vdots \\ \frac{\partial \Phi_d(x)}{\partial x_1} & \cdots & \frac{\partial \Phi_d(x)}{\partial x_d} \end{bmatrix}, \qquad x \in \overline{\mathbb{B}}^d, \tag{7}$$

$$K(s) \equiv (D\Psi)(s) = \begin{bmatrix} \frac{\partial \Psi_1(s)}{\partial s_1} & \cdots & \frac{\partial \Psi_1(s)}{\partial s_d} \\ \vdots & \ddots & \vdots \\ \frac{\partial \Psi_d(s)}{\partial s_1} & \cdots & \frac{\partial \Psi_d(s)}{\partial s_d} \end{bmatrix}, \qquad s \in \overline{\Omega} \tag{8}$$

denote the Jacobian matrix of the transformations Φ and Ψ, respectively. Assume $J(x)$ is nonsingular on $\overline{\mathbb{B}}^d$,

$$\det J(x) \neq 0, \qquad x \in \overline{\mathbb{B}}^d;$$

and without loss of generality, assume

$$\det J(x) > 0, \qquad x \in \overline{\mathbb{B}}^d.$$

Differentiating the identity $\Phi(\Psi(s)) = s$ over Ω, or the identity $\Psi(\Phi(x)) = x$ over $\mathbb{B}^d$, leads to

$$J(x)K(s) = I, \qquad x = \Psi(s), \tag{9}$$

Thus the components of $K(s)$ can be obtained by using

$$K(s) = J(x)^{-1}, \qquad s = \Phi(x). \tag{10}$$

$$K(\Phi(x)) = J(x)^{-1}, \qquad x \in \overline{\mathbb{B}}^d. \tag{11}$$

Let v denote a general function defined over Ω. For the transformation $s = \Phi(x)$, introduce the notation $\widetilde{v}(x) = v(\Phi(x))$; or equivalently, $v(s) = \widetilde{v}(\Psi(s))$. Consider the derivatives with respect to s of $v(s)$. Let ∇_s denote the gradient with respect to the components of s; and do similarly for ∇_x. Then

$$\begin{aligned} \nabla_s v(s) &= K(s)^{\mathrm{T}} \nabla_x \widetilde{v}(x), \qquad x = \Psi(s), \\ \nabla_x \widetilde{v}(x) &= J(x)^{\mathrm{T}} \nabla_s v(s), \qquad s = \Phi(x), \end{aligned} \tag{12}$$

with $\nabla_x \widetilde{v}(x)$ the gradient of $\widetilde{v}(x)$ written as a column vector, and analogously for $\nabla_s v(s)$. Further derivatives are considered later in §3.1.

2.1 Approximation space

For $\Omega = \mathbb{B}^d$, introduce the approximation space

$$\widetilde{\mathscr{X}_n} = \left\{ \left(1 - |x|^2\right)^2 p(x) \mid p \in \Pi_n^d \right\},$$

where Π_n^d denotes the space of all polynomials in d variables and of degree $\le n$. For general Ω, use the approximation space

$$\mathscr{X}_n = \left\{ \chi \circ \Phi^{-1} \mid \chi \in \widetilde{\mathscr{X}_n} \right\}.$$

Let $\mathscr{V}_k$ denote the space of all polynomials of degree k that are orthogonal in $L^2\left(\mathbb{B}^d\right)$ to all polynomials in Π_{k-1}^d using the standard inner product

$$(f, g) = \int_{\mathbb{B}^d} f(x)\, g(x)\, dx.$$

More precisely,

$$\mathscr{V}_k = \left\{ p \in \Pi_k^d \mid (p,q) = 0, \quad \forall q \in \Pi_{k-1}^d \right\}, \qquad k = 1, 2, \ldots,$$

and $\mathscr{V}_0$ is the set of all constant functions. Then

$$\Pi_n^d = \mathscr{V}_0 \oplus \mathscr{V}_1 \oplus \cdots \oplus \mathscr{V}_n,$$

is an orthogonal decomposition of Π_n^d within $L^2\left(\mathbb{B}^d\right)$. A basis for Π_n^d is defined by first defining a basis for each subspace $\mathscr{V}_k$, $k = 0, 1, \ldots, n$. Let $\left\{\varphi_{k,j}\right\}$ be an orthonormal basis of $\mathscr{V}_k$, let $M_k = \dim \mathscr{V}_k$, and define

$$\chi_{k,j}(x) = \left(1 - |x|^2\right)^2 \varphi_{k,j}(x), \qquad j = 1, \ldots, M_k.$$

Denote the corresponding basis for $\widetilde{\mathscr{X}_n}$ by $\{\chi_\ell(x) \mid 1 \le \ell \le N_n\}$,

$$N_n \equiv M_0 + \cdots + M_n.$$

Let $\left\{\psi_j \mid 1 \le j \le N_n\right\}$ be the corresponding basis for $\mathscr{X}_n$ using $\psi_\ell = \chi_\ell \circ \Phi^{-1}$. Note that for $d = 2$,

$$M_n = n + 1, \qquad N_n = \tfrac{1}{2}(n+1)(n+2).$$

Orthonormal bases $\left\{\varphi_{k,j}\right\}$ for $\mathscr{V}_k$, $k \ge 0$, are considered in [2], [13], [22].

3 The numerical method

The numerical method is a Galerkin method for approximating (4): find $u_n \in \mathscr{X}_n$ for which

$$\mathscr{A}(u_n, v) = (f, v), \qquad \forall v \in \mathscr{X}_n.$$

This is the standard variational framework used with finite element methods, with the approximating elements required to belong to $H_0^2(\Omega)$, a significant requirement.

Write

$$u_n(s) = \sum_{j=1}^{N_n} \alpha_j \psi_j(s).$$

Then the coefficients $\left\{\alpha_j\right\}$ must satisfy the linear system

$$\sum_{j=1}^{N_n} \alpha_j \mathscr{A}(\psi_j, \psi_i) = \ell_f(\psi_i), \qquad i = 1, \ldots, N_n. \tag{13}$$

The Lax-Milgram Theorem (cf. [7, §8.3], [9, §2.7], [10, p. 8]) implies the existence of u_n for all n, with

$$\|u_n\|_2 \le \frac{1}{c_e}\|\ell_f\|.$$

For the error in this Galerkin method, Cea's Lemma (cf. [7, p. 371], [9, p. 62], [10, p. 104]) implies the convergence of u_n to u, and moreover,

$$\|u-u_n\|_2 \le \frac{c_{\mathscr{A}}}{c_e} \inf_{v\in\mathscr{X}_n} \|u-v\|_2. \tag{14}$$

It remains to bound the best approximation error on the right side of this inequality. The error analysis is similar to that given in the earlier papers [1], [4], [6].

To bound the right side, make use of the following connection between norms in $H^k(\Omega)$ and $H^k\left(\mathbb{B}^d\right)$; the proof is omitted.

Lemma 1. *Assume $\Phi \in C^\infty(\Omega)$. Let $v \in H^k(\Omega)$ for some $k \ge 0$, $k \in \mathbb{N}_0$, and let $\widetilde{v}(x) = v(\Phi(x))$. Then*

$$c_{1,k}\|\widetilde{v}\|_{H^k\left(\mathbb{B}^d\right)} \le \|v\|_{H^k(\Omega)} \le c_{2,k}\|\widetilde{v}\|_{H^k\left(\mathbb{B}^d\right)} \tag{15}$$

for some $c_{1,k}, c_{2,k} > 0$ independent of v.

In order to look at rates of convergence as a function of n, this lemma is used to convert the bound (14) to the equivalent bound

$$\|\widetilde{u}-\widetilde{u}_n\|_2 \le c \inf_{\widetilde{v}\in\widetilde{\mathscr{X}_n}} \|\widetilde{u}-\widetilde{v}\|_2, \tag{16}$$

c a generic constant dependent on Φ, but not on u. Assume $u \in H_0^k(\Omega)$, and equivalently $\widetilde{u} \in H_0^k\left(\mathbb{B}^d\right)$, $k \ge 2$. Bounding the right side of (16) using [17, Thm 4.3] leads to the error bound

$$\|\widetilde{u}-\widetilde{u}_n\|_2 \le \frac{c}{n^{k-2}} \|\widetilde{u}\|_{H^k\left(\mathbb{B}^d\right)}. \tag{17}$$

Combined with Lemma 1 and (14),

$$\|u-u_n\|_2 \le \frac{c}{n^{k-2}} \|u\|_{H^k(\Omega)}, \tag{18}$$

again with c a generic constant. To obtain convergence for $k = 2$, it can be shown that

$$\inf_{\widetilde{v}\in\widetilde{\mathscr{X}_n}} \|\widetilde{u}-\widetilde{v}\|_2 \to 0 \quad \text{as} \quad n \to \infty.$$

This follows because the polynomials $\cup_{n\ge 0}\widetilde{\mathscr{X}}_n$ are dense in $H_0^2(\Omega)$ [note the comments following [17, Thm 4.3] and the denseness of the polynomials $\cup_{n\ge 0}\Pi_n^d$ in $H^k\left(\mathbb{B}^d\right)$].

3.1 Evaluating the integrals

The integrals

$$\mathscr{A}(\psi_i,\psi_j)=\int_\Omega\left[\Delta\psi_i(s)\,\Delta\psi_j(s)+\gamma(s)\,\psi_i(s)\,\psi_j(s)\right]ds \tag{19}$$

must be computed. Begin by converting to an integral over $\mathbb{B}^d$ .using the transformation $s=\Phi(x)$:

$$\begin{aligned}\mathscr{A}(\psi_i,\psi_j)=\int_{\mathbb{B}^d}&\Big[\Delta_s\psi_i(s)\big|_{s=\Phi(x)}\,\Delta_s\psi_j(s)\big|_{s=\Phi(x)}\\&+\gamma(\Phi(x))\,\chi_i(x)\,\chi_j(x)\Big]\det J(x)\,dx.\end{aligned}$$

The quantities $\Delta_s\psi_i(s)$, $i=1,\ldots,N_n$, must be converted to functions involving derivatives with respect to x for $\chi_i(x)$.

For the transformation $x=\Psi(s)$, let $v(s)=\widetilde{v}(\Psi(s))$; or equivalently, $\widetilde{v}(x)=v(\Phi(x))$. Look at the derivatives with respect to s of $v(s)$. Then for $i=1,\ldots,d$,

$$\begin{aligned}\frac{\partial v(s)}{\partial s_i}&=\sum_{j=1}^d\frac{\partial\widetilde{v}(x)}{\partial x_j}\bigg|_{x=\Psi(s)}\times\frac{\partial\Psi_j(s)}{\partial s_i}\\&=\left[\frac{\partial\Psi_1}{\partial s_i},\ldots,\frac{\partial\Psi_d}{\partial s_i}\right]\nabla_x\widetilde{v}(x),\qquad x=\Psi(s).\end{aligned}$$

This is a proof of (12). Next,

$$\begin{aligned}\frac{\partial^2 v(s)}{\partial s_i^2}&=\frac{\partial}{\partial s_i}\left[\sum_{j=1}^d\frac{\partial\widetilde{v}(x)}{\partial x_j}\bigg|_{x=\Psi(s)}\times\frac{\partial\Psi_j(s)}{\partial s_i}\right]\\&=\sum_{j=1}^d\frac{\partial\widetilde{v}(x)}{\partial x_j}\bigg|_{x=\Psi(s)}\times\frac{\partial^2\Psi_j(s)}{\partial s_i^2}\\&\quad+\sum_{j=1}^d\frac{\partial\Psi_j(s)}{\partial s_i}\sum_{k=1}^d\frac{\partial^2\widetilde{v}(x)}{\partial x_j\partial x_k}\frac{\partial\Psi_k(s)}{\partial s_i}.\end{aligned}$$

Summing over i,

$$\begin{aligned}\Delta_s v(s)=&\sum_{i,j=1}^d\frac{\partial\widetilde{v}(x)}{\partial x_j}\bigg|_{x=\Psi(s)}\times\frac{\partial^2\Psi_j(s)}{\partial s_i^2}\\&+\sum_{i,j,k=1}^d\frac{\partial\Psi_j(s)}{\partial s_i}\frac{\partial^2\widetilde{v}(x)}{\partial x_j\partial x_k}\frac{\partial\Psi_k(s)}{\partial s_i}.\end{aligned}\tag{20}$$

Look at the terms in (20). First,

$$\begin{aligned}\sum_{i,j=1}^d\frac{\partial\widetilde{v}(x)}{\partial x_j}\bigg|_{x=\Psi(s)}\times\frac{\partial^2\Psi_j(s)}{\partial s_i^2}&=\sum_{j=1}^d\frac{\partial\widetilde{v}(x)}{\partial x_j}\Delta_s\Psi_j(s)\\&=\left[\Delta_s\Psi_1(s),\ldots,\Delta_s\Psi_d(s)\right]\nabla_x\widetilde{v}(x).\end{aligned}\tag{21}$$

Using the notation of (8),

$$\begin{aligned}\sum_{i,j,k=1}^{d}\frac{\partial\Psi_j(s)}{\partial s_i}\frac{\partial^2\widetilde{v}(x)}{\partial x_j\partial x_k}\frac{\partial\Psi_k(s)}{\partial s_i} &= \sum_{j,k=1}^{d}\frac{\partial^2\widetilde{v}(x)}{\partial x_j\partial x_k}\sum_{i=1}^{d}\frac{\partial\Psi_j(s)}{\partial s_i}\frac{\partial\Psi_k(s)}{\partial s_i}\\ &= \sum_{j,k=1}^{d}\frac{\partial^2\widetilde{v}(x)}{\partial x_j\partial x_k}\left[K(s)_{j,*}\right]\left[K(s)_{k,*}\right]^{\mathrm{T}} \qquad (22)\\ &= \sum_{j,k=1}^{d}\frac{\partial^2\widetilde{v}(x)}{\partial x_j\partial x_k}\left[K(s)K(s)^{\mathrm{T}}\right]_{j,k}.\end{aligned}$$

Returning to (20) and combining terms,

$$\begin{aligned}\Delta_s v(s) &= \left[\Delta_s\Psi_1(s),\ldots,\Delta_s\Psi_d(s)\right]\nabla_x\widetilde{v}(x)\\ &\quad+\sum_{j,k=1}^{d}\frac{\partial^2\widetilde{v}(x)}{\partial x_j\partial x_k}\left[K(s)_{j,*}\right]\left[K(s)_{k,*}\right]^{\mathrm{T}}. \qquad (23)\end{aligned}$$

Formula (22) can be evaluated from knowing $J(x)$; see (10) above. The formula (23) is to be evaluated with

$$\widetilde{v}(x)=\chi_\ell(x), \qquad 1\le\ell\le N_n,$$

so as to create the elements $\mathscr{A}(\psi_i,\psi_j)$.

To evaluate (21), we need $\Delta_s\Psi_j(s)$, $1\le j\le d$. The first derivatives of Ψ can be obtained from $D_s\Psi(s)=\left[D_x\Phi(x)\right]^{-1}$ where $s=\Phi(x)$. How to obtain the functions $\Delta_s\Psi_j(s)$? Begin by differentiating

$$s=\Phi(\Psi(s)), \qquad s\in\Omega,$$

or

$$s_j=\Phi_j(\Psi_1(s),\ldots,\Psi_d(s)), \qquad 1\le j\le d.$$

The derivative with respect to s_i yields

$$\delta_{i,j}=\sum_{k=1}^{d}\left.\frac{\partial\Phi_j(x)}{\partial x_k}\right|_{x=\Psi(s)}\times\frac{\partial\Psi_k(s)}{\partial s_i}, \qquad 1\le i,j\le d. \qquad (24)$$

Differentiate the components of (9), given in (24), with respect to s_ℓ: for $1\le i,j,\ell\le d$,

$$0=\sum_{k=1}^{d}\frac{\partial\Phi_j(x)}{\partial x_k}\frac{\partial^2\Psi_k(s)}{\partial s_i\partial s_\ell}+\sum_{k=1}^{d}\frac{\partial\Psi_k(s)}{\partial s_i}\sum_{m=1}^{d}\frac{\partial^2\Phi_j(x)}{\partial x_k\partial x_m}\frac{\partial\Psi_m(s)}{\partial s_\ell}.$$

Let $\ell=i$,

$$
\begin{aligned}
0 &= \sum_{k=1}^{d} \frac{\partial \Phi_j(x)}{\partial x_k} \frac{\partial^2 \Psi_k(s)}{\partial s_i^2} + \sum_{k=1}^{d} \frac{\partial \Psi_k(s)}{\partial s_i} \sum_{m=1}^{d} \frac{\partial^2 \Phi_j(x)}{\partial x_k \partial x_m} \frac{\partial \Psi_m(s)}{\partial s_i}, \\
&= \sum_{k=1}^{d} \frac{\partial \Phi_j(x)}{\partial x_k} \frac{\partial^2 \Psi_k(s)}{\partial s_i^2} + \sum_{k,m=1}^{d} \frac{\partial^2 \Phi_j(x)}{\partial x_k \partial x_m} \frac{\partial \Psi_k(s)}{\partial s_i} \frac{\partial \Psi_m(s)}{\partial s_i}.
\end{aligned}
$$

Sum over i: for $1 \le j \le d$,

$$
\begin{aligned}
0 &= \sum_{k=1}^{d} \frac{\partial \Phi_j(x)}{\partial x_k} \Delta_s \Psi_k(s) + \sum_{k,m=1}^{d} \frac{\partial^2 \Phi_j(x)}{\partial x_k \partial x_m} \sum_{i=1}^{d} \frac{\partial \Psi_k(s)}{\partial s_i} \frac{\partial \Psi_m(s)}{\partial s_i} \\
&= \sum_{k=1}^{d} \frac{\partial \Phi_j(x)}{\partial x_k} \Delta_s \Psi_k(s) + \sum_{k,m=1}^{d} \frac{\partial^2 \Phi_j(x)}{\partial x_k \partial x_m} \left[K(s)_{k,*} \right] \left[K(s)_{m,*} \right]^{\mathrm{T}} \qquad (25) \\
&= \sum_{k=1}^{d} \frac{\partial \Phi_j(x)}{\partial x_k} \Delta_s \Psi_k(s) + \sum_{k,m=1}^{d} \frac{\partial^2 \Phi_j(x)}{\partial x_k \partial x_m} \left[K(s) K(s)^{\mathrm{T}} \right]_{k,m}.
\end{aligned}
$$

Introduce

$$
\Delta_s \Psi(s) = \left[\Delta_s \Psi_1(s), \ldots, \Delta_s \Psi_d(s) \right]^{\mathrm{T}},
$$

$$
D^2 \Phi_j(x) = \begin{bmatrix} \frac{\partial^2 \Phi_j(x)}{\partial x_1 \partial x_1} & \cdots & \frac{\partial^2 \Phi_j(x)}{\partial x_1 \partial x_d} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 \Phi_j(x)}{\partial x_d \partial x_1} & \cdots & \frac{\partial^2 \Phi_j(x)}{\partial x_d \partial x_d} \end{bmatrix}, \qquad 1 \le j \le d.
$$

Introduce the dot product of two arrays of the same dimension:

$$
A \odot B = \sum_{i,j} A_{i,j} B_{i,j}.
$$

Then (25) can be written as

$$
0 = \left[J(x)_{j,*} \right] \left[\Delta_s \Psi(s) \right] + D^2 \Phi_j(x) \odot \left[K(s) K(s)^{\mathrm{T}} \right].
$$

From (25),

$$
\begin{aligned}
0 &= J(x) \Delta_s \Psi(s) + \begin{bmatrix} D^2 \Phi_1(x) \odot \left[K(s) K(s)^{\mathrm{T}} \right] \\ \vdots \\ D^2 \Phi_d(x) \odot \left[K(s) K(s)^{\mathrm{T}} \right] \end{bmatrix} \\
&\equiv J(x) \Delta_s \Psi(s) + \mathscr{D}^2 \Phi(x) \odot \left[K(s)^{\mathrm{T}} K(s) \right].
\end{aligned}
$$

which contains an implicit definition of $\mathscr{D}^2 \Phi$ and an implicit notational extension of the operation $\odot$. Then

$$\Delta_s \Psi(s) = -J(x)^{-1}\left\{\mathscr{D}^2\Phi(x)\odot\left[K(s)K(s)^{\mathrm{T}}\right]\right\},$$

and recall (10) to compute $K(s)$.

This allows computing the coefficients $\mathscr{A}(\psi_i,\psi_j)$ of (13) by means of the change of variables $s=\Phi(x)$. Rewrite (23) as

$$\Delta_s v(s) = \left[\Delta_s \Psi(s)\right]^{\mathrm{T}}\nabla_x \widetilde{v}(x) + D^2\widetilde{v}(x)\odot\left[K(s)K(s)^{\mathrm{T}}\right]. \tag{26}$$

Returning to $\mathscr{A}(\psi_i,\psi_j)$, apply (26) with

$$\begin{aligned}\widetilde{v}(x) = \chi_{n,j}(x) &= \left(1-|x|^2\right)^2\varphi_{n,j}(x)\\ &= \left(1-x_1^2-\cdots-x_d^2\right)^2\varphi_{n,j}(x)\end{aligned}$$

for $1\le j\le M_k$, $0\le k\le n$.

We need to find the first and second order derivatives of $\chi_{n,j}(x)$, and thus also of $\varphi_{n,j}(x)$.

$$\frac{\partial\chi_{n,j}(x)}{\partial x_k} = -4x_k\left(1-x_1^2-\cdots-x_d^2\right)\varphi_{n,j}(x) + \left(1-|x|^2\right)^2\frac{\partial\varphi_{n,j}(x)}{\partial x_k},$$

$$\begin{aligned}\frac{\partial^2\chi_{n,j}(x)}{\partial x_k^2} = &\left\{-4\left(1-x_1^2-\cdots-x_d^2\right)+8x_k^2\right\}\varphi_{n,j}(x)\\ &-8x_k\left(1-x_1^2-\cdots-x_d^2\right)\frac{\partial\varphi_{n,j}(x)}{\partial x_k}\\ &+\left(1-|x|^2\right)^2\frac{\partial^2\varphi_{n,j}(x)}{\partial x_k^2}.\end{aligned}$$

For $\ell\ne k$,

$$\begin{aligned}\frac{\partial^2\chi_{n,j}(x)}{\partial x_k\partial x_\ell} = &\, 8x_k x_\ell\varphi_{n,j}(x)\\ &-4\left(1-x_1^2-\cdots-x_d^2\right)\left\{x_k\frac{\partial\varphi_{n,j}(x)}{\partial x_\ell}+x_\ell\frac{\partial\varphi_{n,j}(x)}{\partial x_k}\right\}\\ &+\left(1-|x|^2\right)^2\frac{\partial^2\varphi_{n,j}(x)}{\partial x_k\partial x_\ell}.\end{aligned}$$

These can be combined with (26) to compute $\Delta_s\chi_j$ and thus to compute $\mathscr{A}(\psi_i,\psi_j)$ for $1\le i,j\le N_n$.

The next step is to look at particular orthonormal polynomials $\left\{\varphi_{n,j}(x)\right\}$ and to compute

$$\begin{aligned} &\varphi_{n,j}, && 1 \le j \le M_n, \\ &\frac{\partial \varphi_{n,j}(x)}{\partial x_k}, && \begin{array}{c} 1 \le j \le M_n \\ 1 \le k \le d \end{array}, \\ &\frac{\partial^2 \varphi_{n,j}(x)}{\partial x_k \partial x_\ell}, && \begin{array}{c} 1 \le j \le M_n \\ 1 \le k, \ell \le d \end{array}. \end{aligned}$$

The best choice as regards speed of calculation is to use the polynomials discussed in [2], as they satisfy a triple recursion that allows for a rapid calculation. For the planar case, these are given by

$$\varphi_{n,k}(x) = \frac{1}{h_{k,n}} C_{n-k}^{k+1}(x_1) \left(1 - x_1^2\right)^{\frac{k}{2}} C_k^{\frac{1}{2}} \left(\frac{x_2}{\sqrt{1 - x_1^2}} \right), \qquad x \in \mathbb{B}^2, \tag{27}$$

for $k = 0, \ldots, n$, $n = 0, 1, \ldots$ The quantity $C_m^{\lambda}(t)$, $m \ge 0$, denotes the Gegenbauer polynomial of degree m and index λ.

For the three dimensional case, we use the polynomials

$$\begin{aligned} \varphi_{n,j,k}(x) = {} & \frac{1}{h_{j,k}} C_{n-j-k}^{j+k+3/2}(x_1)(1-x_1^2)^{j/2} \ldots \\ & \times C_j^{k+1}\left(\frac{x_2}{\sqrt{1-x_1^2}}\right)(1-x_1^2-x_2^2)^{k/2} C_k^{1/2}\left(\frac{x_3}{\sqrt{1-x_1^2-x_2^2}}\right), \\ & x \in \mathbb{B}^3, \, 0 \le j+k \le n, \, n = 0, 1, \ldots \end{aligned} \tag{28}$$

which uses again the Gegenbauer polynomials. The numbers $h_{j,k}$ are normalization constants, see [13], and see [2] for the triple recursion.

4 Numerical Examples

4.1 Planar Examples

Our first examples are for Ω a planar region, and thus $\Phi : \mathbb{B}^2 \to \Omega$.

Example 1. Begin with the elliptical region Ω defined by the mapping $s = \Phi(x)$, $x \in \mathbb{B}^2$,

$$\begin{aligned} s_1 &= 2x_1 + x_2 \\ s_2 &= 3x_1 - 4x_2. \end{aligned} \tag{29}$$

Choose

$$f(s) = 10 \cos(s_1 - 0.1) \sin(s_2 + 0.1) \tag{30}$$

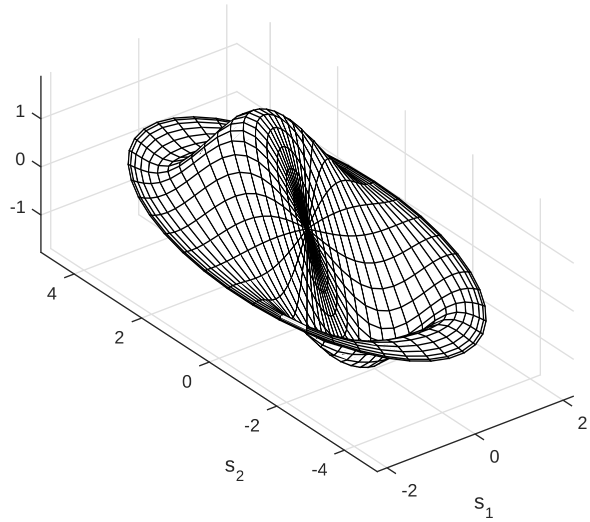


Fig. 1 Solution u with f given by (30), $\gamma(s) \equiv 1$, and the region (29)

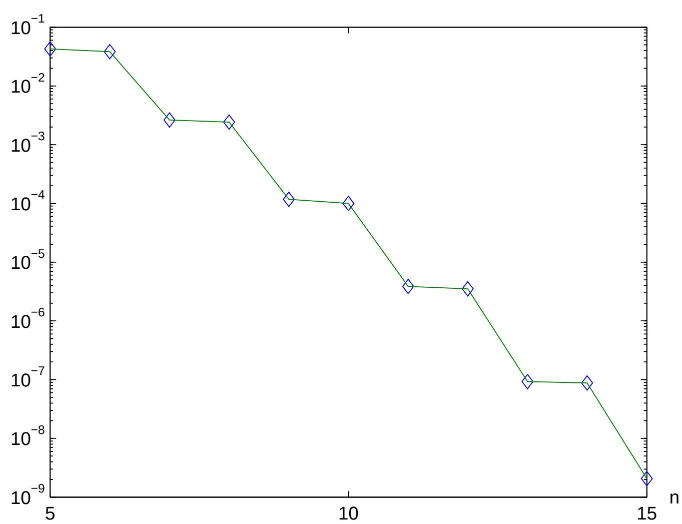


Fig. 2 Computed error in u_n with f given by (30) and $\gamma(s) \equiv 1$.

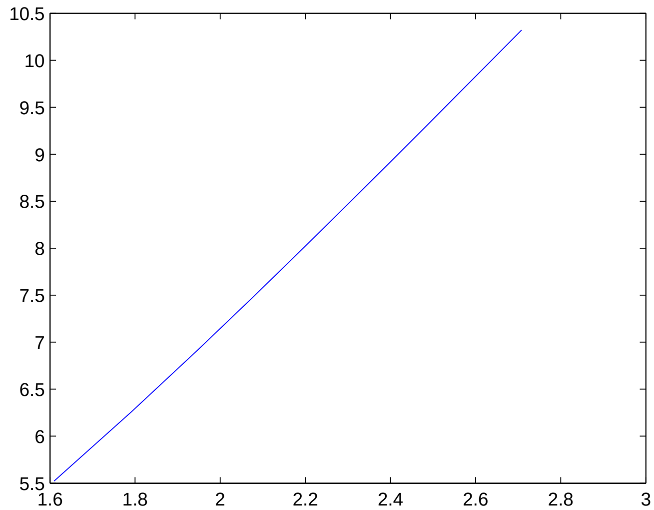


Fig. 3 $\log n$ vs. $\log(cond)$, with *cond* the condition number of (13), with the region (29)

and $\gamma(s) \equiv 1$ over Ω. The solution is shown in Figure 1. The true solution is unknown, so the error is estimated by using u_{n^*} as the 'true' solution with n^* much larger than n being used. In the present case, $n^* = 20$ was used. The maximum errors are shown in Figure 2, and it appears to be an exponential decrease in the error. Figure 3 is a graph of $\log n$ vs. $\log(cond)$, with *cond* the condition number of (13). It indicates that the condition number is $\mathscr{O}(n^p)$ for some power p; experimentally and roughly, $p \approx 4.5$, and $p = 4$ seems most likely to be the theoretical power. That would be consistent with the condition number being $\mathscr{O}\left(N_n^2\right)$, as was observed earlier with the spectral method for the Neumann boundary value problem for second order equations.

Example 2. Consider the boundary mapping

$$\begin{aligned} \varphi(\theta) &= \rho(\theta)(\cos\theta, \sin\theta), \\ \rho(\theta) &= 3 + \cos\theta + 2\sin\theta, \qquad 0 \le \theta \le 2\pi. \end{aligned} \tag{31}$$

This can be extended to a polynomial mapping of degree 2 in various ways, as discussed in [5], and one such mapping is illustrated in Figure 4. This mapping Φ is obtained using (1) the interpolation/quadrature method of §3 in [5], followed by (2) computing the least squares polynomial approximation over $\mathbb{B}^2$ of degree 2 in each component.

The equation (1) is solved with the same choices for γ and f as in Example 1. Figure 5 illustrates the solution, using u_{20}. The errors are shown in Figure 6.

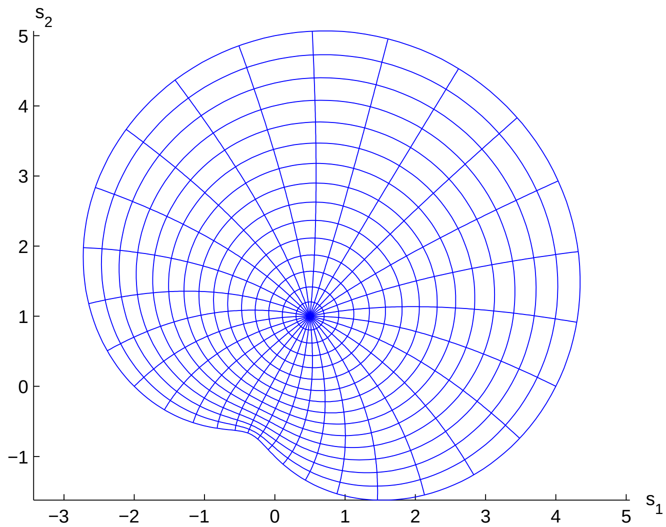


Fig. 4 Mappings for limacon boundary mapping (31)

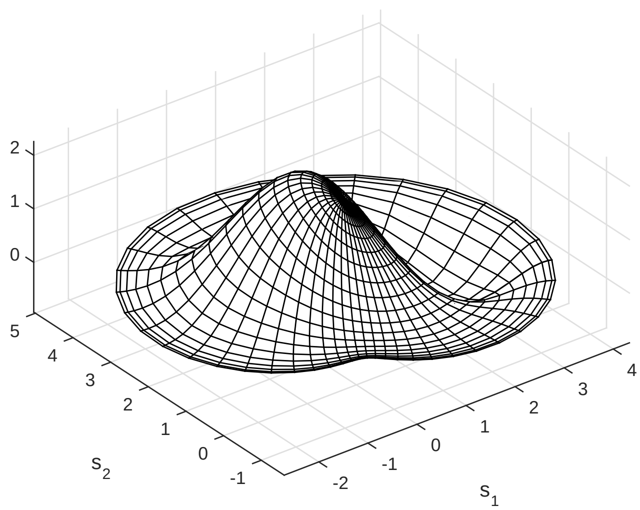


Fig. 5 Solution u with f given by (30), $\gamma(s) \equiv 1$, and $\partial\Omega$ given by (31)

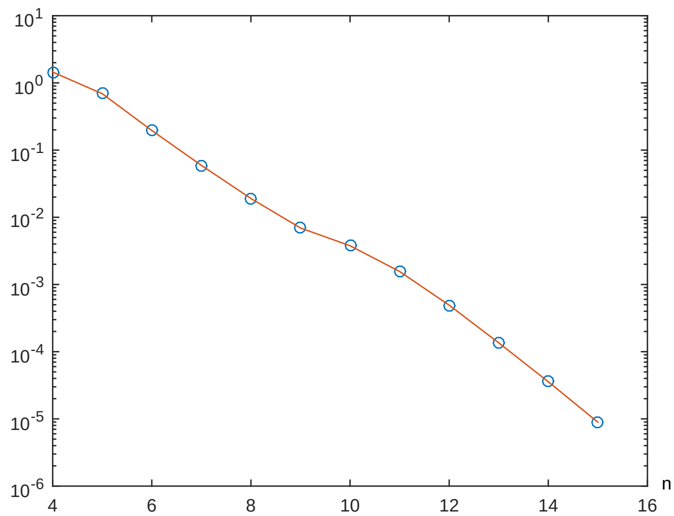


Fig. 6 Computed error in u_n with f given by (30), $\gamma(s) \equiv 1$, and the boundary mapping (31)

The condition numbers, shown in Figure 7, appear to increase like $\mathscr{O}\left(N_n^2\right)$, as with Example 1.

Example 3. Consider the mapping

$$\Phi_1(x) = \left[x_1 - x_2 + ax_1^2, x_1 + x_2\right]^{\mathrm{T}}, \qquad x \in \overline{\mathbb{B}}^2, \tag{32}$$

for a given $0 < a < 1$, with the image defining Ω. In addition, use the interpolation/quadrature method of §3 in [5] to create another mapping Φ_2 that agrees with Φ_1 on the boundary of $\mathbb{B}^2$. These mappings are illustrated in Figure 8. Clearly Φ_2 is a 'better behaved' mapping as compared to Φ_1. We solve $\Delta^2 u + \gamma u = f$ as before, but now let

$$f(s) = 200\cos(st)\sin(t + 0.1). \tag{33}$$

The solution is shown in Figure 9. The maximum errors are shown in Figure 10, and there appears to be an exponential decrease in the error. The condition numbers are shown in Figure 11, and again they appear to increase like $\mathscr{O}\left(N_n^2\right)$.

4.2 A three dimensional example

Example 4. Here we consider the case of an ellipsoid

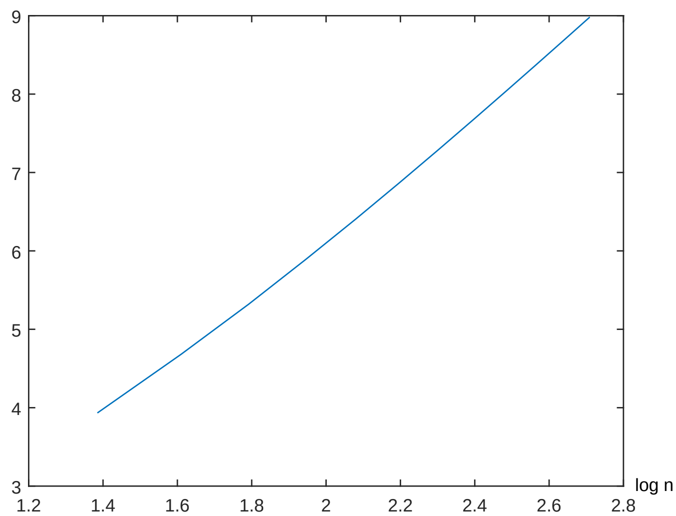
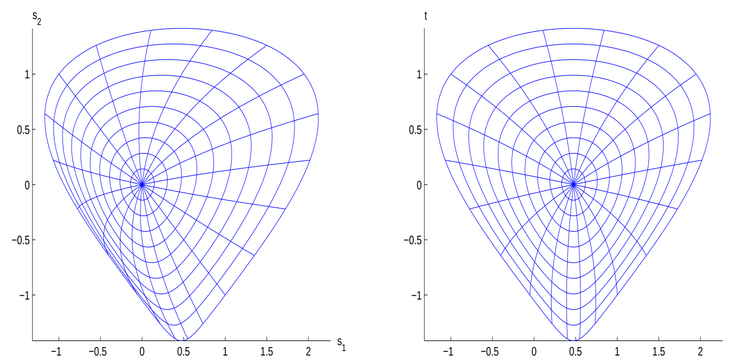


Fig. 7 $\log n$ vs. $\log(cond)$, with *cond* the condition number of (13), with the boundary mapping (31).



(a) Φ_1 (b) Φ_2

Fig. 8 The mapping Φ for boundary (32) with a=0.95

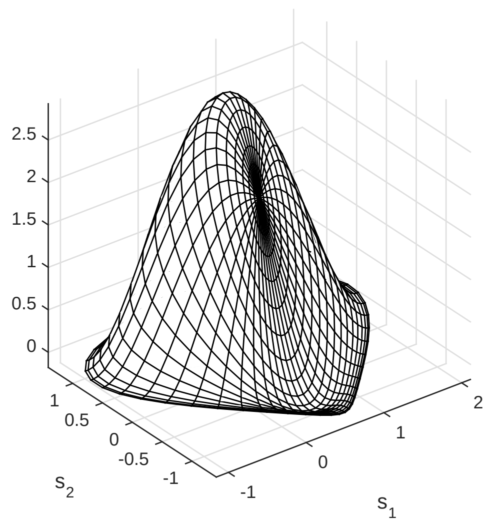


Fig. 9 Solution u with f given by (33), $\gamma(s) \equiv 1$, and $\partial\Omega$ given by (32)

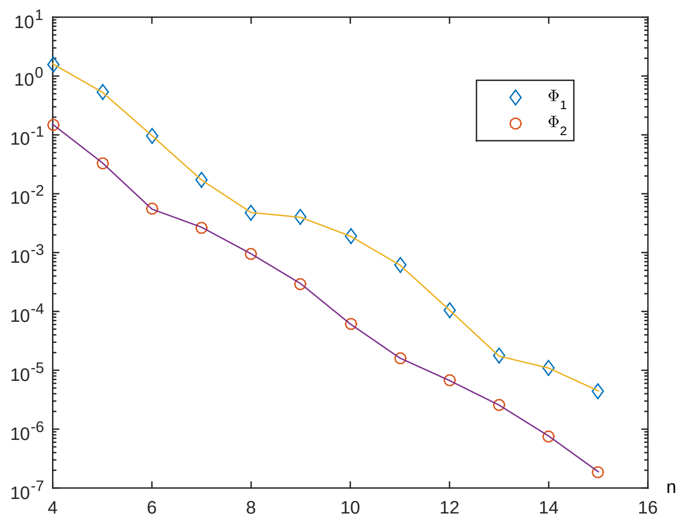


Fig. 10 Computed error in u_n with f given by (33), for the mappings Φ_1 and Φ_2 with the boundary specified by (32).

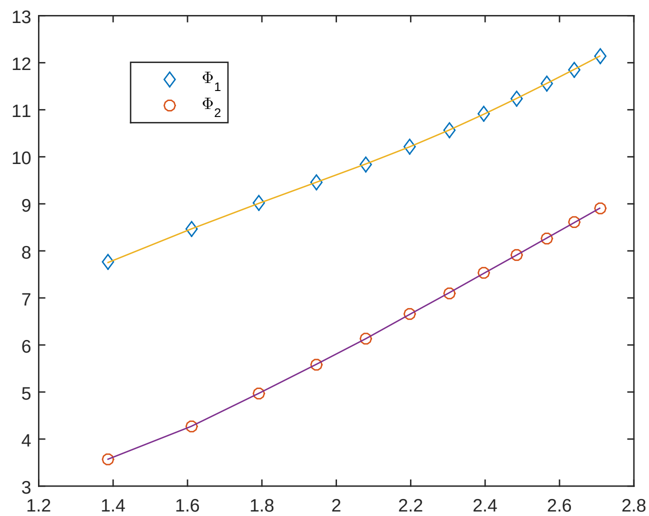


Fig. 11 $\log n$ vs. $\log(cond)$, with *cond* the condition number of (13), for the mappings Φ_1 and Φ_2 with the boundary specified by (32).

$$\Omega = \{(s_1,s_2,s_3) \,|\, s_1^2 + \left(\frac{s_2}{3}\right)^2 + \left(\frac{s_3}{2}\right)^2 \leq 1\} \tag{34}$$

with the obvious mapping

$$\Phi(x_1,x_2,x_3) = [x_1,3x_2,2x_3],\ [x_1,x_2,x_3] \in \mathbb{B}^3. \tag{35}$$

We solve equation (1) with $\gamma(s) \equiv 1$ and calculate the right hand side f_1 in such a way that the solution of (1) is given by

$$u(s_1,s_2,s_3) = \left(1 - s_1^2 - \left(\frac{s_2}{3}\right)^2 - \left(\frac{s_3}{2}\right)^2\right)^2 e^{3(s_1+s_2/3+s_3/2)}. \tag{36}$$

To study the influence of faster growing derivatives we use a second right hand side f_2 on the same domain Ω, such that the solution is given

$$v(s_1,s_2,s_3) = \left(1 - s_1^2 - \left(\frac{s_2}{3}\right)^2 - \left(\frac{s_3}{2}\right)^2\right)^2 e^{7(s_1+s_2/3+s_3/2)}. \tag{37}$$

We expect slower but still exponential convergence for the second example. This is confirmed in the numerical calculation, see Figure 12, where the maximum errors are plotted versus n. The error graph for the solution u shows some saturation around $n = 22$, because we reach the precision limit of the Gauß–quadratures that we use for the evaluation of the integrals in equation (13). The graph of $\log(cond)$ versus

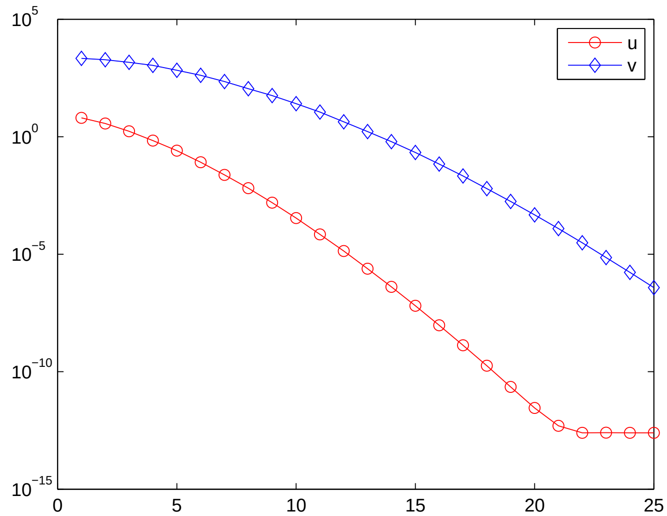


Fig. 12 Computed error in u_n and v_n, for the solutions u and v given in (36) and (37).

$\log n$ in Figure 13 shows again a polynomial behavior. From the numerical results we estimate a condition number of $\mathcal{O}\left(N_n^2\right)$, where we remember that $N_n = \mathcal{O}(n^3)$.

5 Nonhomogeneous boundary conditions

Consider the Dirichlet biharmonic problem

$$\begin{aligned} \Delta^2 u(s) + \gamma(s)\, u(s) = f(s), \qquad & s \in \Omega, \\ u(s) = g_1(s), \quad \frac{\partial u(s)}{\partial n_s} = g_2(s), \quad & s \in \partial\Omega. \end{aligned} \tag{38}$$

This can be reduced to two simpler problems. Consider first the standard Dirichlet biharmonic problem

$$\begin{aligned} \Delta^2 w(s) = 0, \qquad & s \in \Omega, \\ w(s) = g_1(s), \quad \frac{\partial w(s)}{\partial n_s} = g_2(s), \quad & s \in \partial\Omega. \end{aligned} \tag{39}$$

Define $v = u - w$. Then v satisfies

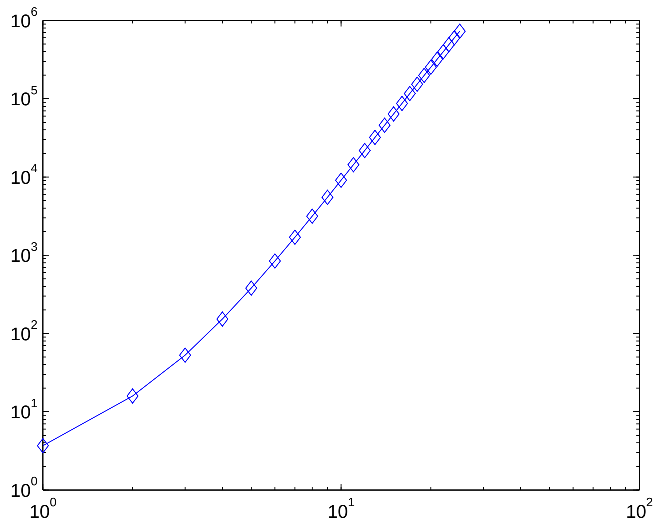


Fig. 13 $\log n$ vs. $\log(cond)$, with *cond* the condition number of (13), with $\gamma(s) \equiv 1$, and the mapping (35) for the domain (34).

$$\begin{aligned} \Delta^2 v(s) + \gamma(s) v(s) = f(s) - \gamma(s) w(s) \equiv \widetilde{f}(s), \qquad s \in \Omega, \\ v(s) = \frac{\partial v(s)}{\partial n_s} = 0, \quad s \in \partial\Omega. \end{aligned} \tag{40}$$

Begin by solving (39) numerically, obtaining an approximating solution $\widehat{w}(s) \approx w(s)$. Then solve (40) with $\widehat{w}(s)$ replacing $w(s)$ in the definition of $\widetilde{f}(s)$. The problem (40) can be solved using the methods given earlier in this paper. Solve for an approximating solution $v_n(s) \approx v(s)$, and then define

$$\widehat{u}(s) = v_n(s) + \widehat{w}(s), \qquad s \in \Omega,$$

as the approximating solution of (38).

To solve (39), a number of methods have been proposed, often using boundary integral equation reformulations. For a review of some of these, see [14, Chaps. 9,15], [15], [16].

Remark. The eigenvalue problem for the biharmonic equation (1)-(2) is discussed and illustrated in the book [3, Chap. 9].

Traditional spectral methods use univariate approximations with a decomposition of the partial differential equation into univariate problems. Consider, for example, using a polar coordinates decomposition of the unit disk. But this leads to problems when treating the solution u at the center of the disk. The present spectral method makes use of the recently developed theory and tools for multivariate approximation over $\mathbb{B}^d$, avoiding artificial problems that can occur when using univariate approximations.

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