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The Numerical Stability of Barycentric Lagrange Interpolation*

Nicholas J. Higham†

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Abstract

The Lagrange representation of the interpolating polynomial can be rewritten in two more computationally attractive forms: a modified Lagrange form and a barycentric form. We give an error analysis of the evaluation of the interpolating polynomial using these two forms. The modified Lagrange formula is shown to be backward stable. The barycentric formula has a less favourable error analysis, but is forward stable for any set of interpolating points with a small Lebesgue constant. Therefore the barycentric formula can be significantly less accurate than the modified Lagrange formula only for a poor choice of interpolating points. This analysis provides further weight to the argument of Berrut and Trefethen that barycentric Lagrange interpolation should be the polynomial interpolation method of choice.

Key words. polynomial interpolation, Lagrange interpolation, barycentric formula, rounding error analysis, backward error, forward error, Lebesgue constant

AMS subject classifications. 65D05, 65G05

1 Introduction

The Lagrange polynomial interpolation formula is widely regarded as being of mainly theoretical interest, as reference to almost any numerical analysis textbook reveals. Yet several authors, including Henrici [5], Rutishauser [9], Salzer [10], Werner [11] and Winrich [12], have noted that certain variants of the Lagrange formula are indeed of practical use. Berrut and Trefethen [1] have recently collected and explained the attractive features of two modified Lagrange formulas. They argue convincingly that interpolation via a barycentric Lagrange formula ought to be the standard method of polynomial interpolation. A question raised but not answered by Berrut and Trefethen is the effect of rounding errors on the two formulas. The purpose of this work, which was begun after reading a draft of [1], is to answer this question.

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We begin, in Section 2, by deriving a condition number for polynomial interpolation. Error analyses of the modified Lagrange formula and barycentric formula are given in Sections 3 and 4, respectively. The modified Lagrange formula is shown to be backward stable and the barycentric formula forward stable for sets of interpolation points with a small Lebesgue constant. We give numerical experiments to illustrate the potential difference in accuracy of the two formulas in Section 5 and then present conclusions in Section 6.

Ours is not the first numerical investigation of the modified Lagrange formula. Rack and Reimer [7] give a rounding error analysis that concludes with a weaker bound than (6) below and they do not identify the backward stability of the formula.

We restrict our attention to the effect of rounding errors on barycentric interpolation. For full details of the many interesting properties of barycentric interpolation the reader should consult Berrut and Trefethen [1].

2 Condition Number

We are interested in the problem of finding the polynomial $p_n(x)$ of degree at most n that interpolates to the data f_j at the distinct points x_j, $j = 0\colon n$. We consider fixed interpolation points x_j, a fixed evaluation point x, and a varying vector f. We will therefore also denote $p_n(x)$ by $p_f(x)$. Inequalities between vectors are interpreted componentwise.

To aid the interpretation of our error bounds we need to define and evaluate a condition number for p_n. Recall that the Lagrange form of $p_n(x)$ is

$$p_n(x) = \sum_{j=0}^{n} f_j \ell_j(x), \qquad \ell_j(x) = \frac{\displaystyle\prod_{k=0,\ k\neq j}^{n} (x - x_k)}{\displaystyle\prod_{k=0,\ k\neq j}^{n} (x_j - x_k)}. \tag{1}$$

Definition 2.1 *The condition number of p_n at x with respect to f is*

$$\mathrm{cond}(x, n, f) = \lim_{\epsilon \to 0} \sup \left\{ \left| \frac{p_f(x) - p_{f+\Delta f}(x)}{\epsilon p_f(x)} \right| : |\Delta f| \leq \epsilon |f| \right\}.$$

In the notation $\mathrm{cond}(x, n, f)$ the term "n" indicates the dependence of cond on the points x_j.

Lemma 2.2

$$\mathrm{cond}(x, n, f) = \frac{\sum_{j=0}^{n} |\ell_j(x) f_j|}{|p_n(x)|} \geq 1, \tag{2}$$

and for any Δf with $|\Delta f| \leq \epsilon |f|$ we have

$$\frac{|p_f(x) - p_{f+\Delta f}(x)|}{|p_f(x)|} \leq \mathrm{cond}(x, n, f) \epsilon.$$

Proof. From

$$p_f(x) - p_{f+\Delta f}(x) = \sum_{j=0}^{n} \ell_j(x) \Delta f_j$$

it is immediate that the claimed expression is an upper bound for $\mathrm{cond}(x, n, f)$, and it is clearly at least 1. Equality is attained for $\Delta f_j = \epsilon\, \mathrm{sign}(\ell_j(x))|f_j|$. The inequality follows trivially. □

3 Modified Lagrange Formula

A trivial rewriting of (1) is

$$p_n(x) = \ell(x) \sum_{j=0}^{n} \frac{w_j}{x - x_j} f_j, \tag{3}$$

where

$$\ell(x) = \prod_{j=0}^{n} (x - x_j)$$

and

$$w_j = \frac{1}{\prod_{k \neq j} (x_j - x_k)}. \tag{4}$$

This is called the "first form of the barycentric interpolation formula" by Rutishauser [9].

For our error analysis we use the standard model of floating point arithmetic [6, Sec. 2.2]:

$$fl(x \mathbin{\mathrm{op}} y) = (x \mathbin{\mathrm{op}} y)(1+\delta)^{\pm 1}, \qquad |\delta| \le u, \quad \mathrm{op} = +, -, *, /,$$

where u is the unit roundoff. Our bounds are expressed in terms of the constant

$$\gamma_k = \frac{ku}{1 - ku}.$$

We also employ the relative error counter, $\langle k \rangle$:

$$\langle k \rangle = \prod_{i=1}^{k} (1+\delta_i)^{\rho_i}, \qquad \rho_i = \pm 1, \quad |\delta_i| \le u. \tag{5}$$

For clarity, we will write $\langle k \rangle_j$ to denote that the k rounding errors in question depend on j. We will use the fact that $|\langle k \rangle - 1| \le \gamma_k = ku/(1 - ku)$ [6, Lem. 3.1]. Finally, we assume that the x_i, f_i and x are floating point numbers.

Lemma 3.1 *The computed weights* $\widehat{w}_j$ *satisfy*

$$\widehat{w}_j = w_j \langle 2n \rangle, \qquad j = 0{:}\, n,$$

while the computed $\widehat{\ell}(x)$ *satisfies*

$$\widehat{\ell}(x) = \ell(x) \langle 2n+1 \rangle.$$

Proof.

$$\begin{aligned} fl\left(\prod_{k\neq j}(x_j - x_k)\right) &= fl\left(\prod_{k\neq j} fl(x_j - x_k)\right) \\ &= fl\left(\prod_{k\neq j}(x_j - x_k)\langle 1\rangle\right) \\ &= \langle n-1\rangle\langle n\rangle \prod_{k\neq j}(x_j - x_k) \\ &= \langle 2n-1\rangle \prod_{k\neq j}(x_j - x_k), \end{aligned}$$

where n rounding errors come from additions and $n-1$ from multiplications. The final division contributes one further rounding error. The expression for $\widehat{\ell}$ is derived similarly, since ℓ involves $n+1$ subtractions and n multiplications. $\square$

Theorem 3.2 *The computed* $\widehat{p}_n(x)$ *from* (3) *satisfies*

$$\widehat{p}_n(x) = \ell(x) \sum_{j=0}^{n} \frac{w_j}{x - x_j} f_j \langle 5n+5\rangle_j .$$

Proof. We have

$$\widehat{p}_n(x) = \widehat{\ell}(x)\langle 1\rangle \sum_{j=0}^{n} \frac{\widehat{w}_j}{x - x_j} f_j \langle 3\rangle_j \langle n\rangle_j ,$$

where the factor $\langle 3\rangle$ accounts for the subtraction in the denominator, the division and the multiplication, and the factor $\langle n\rangle$ accounts for the errors in summation, no matter which ordering is used [6, Chap. 4]. Using Lemma 3.1 we obtain

$$\widehat{p}_n(x) = \ell(x)\langle 2n+2\rangle \sum_{j=0}^{n} \frac{w_j \langle 2n\rangle_j}{x - x_j} f_j \langle 3\rangle_j \langle n\rangle_j ,$$

which yields the result on collecting the rounding error terms. $\square$

This is an extremely strong result: it says that $\widehat{p}_n(x)$ is the exact value at x of the interpolant of a perturbed problem in which the perturbations are small relative changes in the data f. In other words, formula (3) is a backward stable means of evaluating $p_n(x)$. We can hardly expect better: these errors are of the same form, and only $O(n)$ times larger than, the errors in rounding the f_j to floating point form.

A forward error bound follows trivially from Lemma 2.2:

$$\frac{|p_n(x) - \widehat{p}_n(x)|}{|p_n(x)|} \leq \gamma_{5n+5}\,\mathrm{cond}(x, n, f). \tag{6}$$

It is easy to see that there exist rounding errors such that this bound is approximately attained. Theorem 3.2 and (6) justify the use of the formula (3) by Dutt, Gu and Rokhlin [4].

If the x_j or x are not floating point numbers then there can be large relative errors in the differences $fl(fl(x_j) - fl(x_k))$ and $fl(fl(x) - fl(x_j))$. However, the computed $\widehat{p}_n(x)$ can nevertheless be interpreted as the exact result corresponding to slightly perturbed x and points x_j (namely, the rounded values) and slightly perturbed points f_j; so if $\widehat{p}_n(x)$ has a large relative error then the problem itself must be ill conditioned with respect to variations in x and the x_j and f_j.

4 Barycentric Formula

The function values $f_i \equiv 1$ are obviously interpolated by $p_n(x) = 1$, and hence (3) gives $1 = \ell(x) \sum_{j=0}^{n} w_j/(x - x_j)$. Using this equation to eliminate $\ell(x)$ in (3) yields

$$p_n(x) = \frac{\displaystyle\sum_{j=0}^{n} \frac{w_j}{x - x_j} f_j}{\displaystyle\sum_{j=0}^{n} \frac{w_j}{x - x_j}}, \tag{7}$$

which is called the "second (proper) form of the barycentric formula" by Rutishauser [9].

Since this formula is obtained by using a mathematical identity that does not necessarily hold in floating point arithmetic, this second formula might be expected to have different stability properties to the first.

Working in the same way as in the proof of Theorem 3.2, we find that

$$\begin{aligned}
\widehat{p}_n(x) &= \frac{\displaystyle\sum_{j=0}^{n} \frac{w_j \langle 2n \rangle_j}{x - x_j} f_j \langle n+3 \rangle_j}{\displaystyle\sum_{j=0}^{n} \frac{w_j \langle 2n \rangle_j}{x - x_j} \langle n+2 \rangle_j} \langle 1 \rangle \\
&= \frac{\displaystyle\sum_{j=0}^{n} \frac{w_j}{x - x_j} f_j \langle 3n+4 \rangle_j}{\displaystyle\sum_{j=0}^{n} \frac{w_j}{x - x_j} \langle 3n+2 \rangle_j}.
\end{aligned}$$

This result does not admit any useful interpretation in terms of backward error. But it does lead readily to a forward error bound, which is stated in the next result. We note first that the equality (2) can be rewritten

$$\operatorname{cond}(x, n, f) = \frac{\displaystyle\sum_{j=0}^{n} \left| \frac{w_j f_j}{x - x_j} \right|}{\displaystyle\left| \sum_{j=0}^{n} \frac{w_j f_j}{x - x_j} \right|}.$$

Theorem 4.1 *The computed $\widehat{p}_n(x)$ from (7) satisfies*

$$
\begin{aligned}
\frac{|p_n(x)-\widehat{p}_n(x)|}{|p_n(x)|} &\leq (3n+4)u\frac{\sum_{j=0}^n \left|\frac{w_j}{x-x_j}f_j\right|}{\left|\sum_{j=0}^n \frac{w_j}{x-x_j}f_j\right|}+(3n+2)u\frac{\sum_{j=0}^n \left|\frac{w_j}{x-x_j}\right|}{\left|\sum_{j=0}^n \frac{w_j}{x-x_j}\right|}+O(u^2)\\
&= (3n+4)u\,\mathrm{cond}(x,n,f)+(3n+2)u\frac{\sum_{j=0}^n \left|\frac{w_j}{x-x_j}\right|}{\left|\sum_{j=0}^n \frac{w_j}{x-x_j}\right|}+O(u^2) \qquad (8)\\
&= (3n+4)u\,\mathrm{cond}(x,n,f)+(3n+2)u\,\mathrm{cond}(x,n,1)+O(u^2), \qquad (9)
\end{aligned}
$$

where the argument "1" denotes the function with constant value 1. *There exist rounding errors such that this bound is approximately attained.*

We see from (8) that the forward error bound for the barycentric formula contains an extra term not present in that for the first formula: a term that measures the amount of cancellation in the denominator. Since the denominator is independent of f, it is clear that for suitable choices of f and the x_j, the bound (8) can be arbitrarily larger than $\mathrm{cond}(x,n,f)u$. For example, if we take $f_1=1$ and $f_j=0$ for $j>1$, then $\mathrm{cond}(x,n,f)=1$, while for suitable choice of the x_j the second term in (8) can be arbitrarily large. However, from (9) we see that the error bound is significantly larger than that for the modified Lagrange formula only if $\mathrm{cond}(x,n,1)\gg\mathrm{cond}(x,n,f)$: a circumstance that intuitively seems unlikely.

To gain more insight, we assume that the points x_j lie in $[-1,1]$ and express the bound in terms of Λ_n, the Lebesgue constant associated with the points x_j, defined by [8, Chap. 4]

$$\Lambda_n=\sup_{f\in C([-1,1])}\frac{\|P_nf\|}{\|f\|},$$

where P_n is the operator mapping f to its interpolating polynomial at the x_j, $\|f\|=\max_{x\in[-1,1]}|f(x)|$, and $C([-1,1])$ is the space of all continuous functions on $[-1,1]$. It can be shown that [3, Chap. 2]

$$\Lambda_n=\sup_{x\in[-1,1]}\sum_{j=0}^n|\ell_j(x)|.$$

Noting that $\mathrm{cond}(x,n,1)=\sum_{j=0}^n|\ell_j(x)|$, we can weaken (9) to obtain the following result.

Corollary 4.2 *The computed $\widehat{p}_n(x)$ from (7) satisfies*

$$\frac{|p_n(x)-\widehat{p}_n(x)|}{|p_n(x)|}\leq(3n+4)u\,\mathrm{cond}(x,n,f)+(3n+2)u\,\Lambda_n+O(u^2). \qquad (10)$$

For the Chebyshev points of the first kind (the zeros of the degree $n+1$ Chebyshev polynomial) and the Chebyshev points the second kind (the extreme points of the degree n Chebyshev polynomial),

$$\Lambda_n\leq\frac{2}{\pi}\log(n+1)+1.$$

For other "good" sets of points, Λ_n is also slowly growing. For equally spaced points, Λ_n grows exponentially at a rate proportional to $2^n/(n\log n)$. For details of these results see

Brutman [2] or Cheney and Light [3, Chap. 3]. We can conclude that while the barycentric formula is not forward stable in general, it can be significantly less accurate than the modified Lagrange formula only for a poor choice of interpolating points and special f. More specifically, for both sets of Chebyshev points the barycentric formula is guaranteed to be forward stable—that is, it produces relative errors bounded by $g(n)u\mathrm{cond}(x,n,f)$, with g a slowly growing function of n.

The barycentric formula has two practical advantages over the modified Lagrange formula noted by Berrut and Trefethen. First, since the w_j appear linearly in both the numerator and denominator they can be rescaled ($w_j \leftarrow \alpha w_j$) to avoid overflow and underflow; see [1] for a suggested general scaling. Second, for both sets of Chebyshev points, simple explicit formulas are known for the w_j [1], [10].

5 Numerical Experiments

We report an experiment whose purpose is to verify the conclusions of the error analysis and also to provide a comparison between the formulas analyzed here and the Newton divided difference form. The computations were performed in MATLAB, for which $u \approx 10^{-16}$.

We take 30 equally spaced points x_j on $[-1,1]$ (thus $n = 29$) and set $f_j = 0$ for $j = 0\colon n$ and f_n=1. We evaluate the interpolant at 100 equally spaced points on $[-1 + 10^3\epsilon, 1 - 10^3\epsilon]$, where $\epsilon = 2u$ (MATLAB's `eps`). The "exact" values were obtained by computing in high precision using MATLAB's Symbolic Math Toolbox. Figure 1 plots the errors for the modified Lagrange formula, (3), the barycentric formula, (7), and the Newton divided difference form, with the latter form evaluated by nested multiplication. In this figure the x_j are in increasing order. In Figure 2 the x_j have been re-ordered to be in decreasing order.

In this example, $\mathrm{cond}(x,n,f) \equiv 1$, so a forward stable method should give a computed $\widehat{p}_n(x)$ with relative error of order u. In Figure 1 we see that the modified Lagrange formula, (3), performs stably, as it must do in view of our error analysis. The barycentric formula, (7), performs unstably, and given that $\Lambda_n = 3 \times 10^6$, we see that the bound (10) is fairly sharp at the ends of the interval. The same comments apply to Figure 2 (note the different scales on the y-axes). The Newton divided difference formula performs stably in Figure 1 but very unstably in Figure 2.

Finally, to balance this very extreme example we give a more typical one. This example differs from the first only in that the function values come from the Runge function $f(x) = 1/(1 + 25x^2)$ and the points x_j are the Chebyshev points of the first kind. Here, $\max_x \mathrm{cond}(x,n,f) = 7.7$. As Figure 3 shows, both the modified Lagrange formula and the barycentric formula behave in a forward stable way, while the Newton divided difference formula becomes very unstable as x decreases from 0 to -1.

6 Conclusions

The modified Lagrange formula (3) for polynomial interpolation is backward stable with respect to perturbations in the function values. The barycentric formula (7) is not backward stable, but it is forward stable for any set of interpolating points with a small Lebesgue constant, which roughly means points that are clustered towards the end of

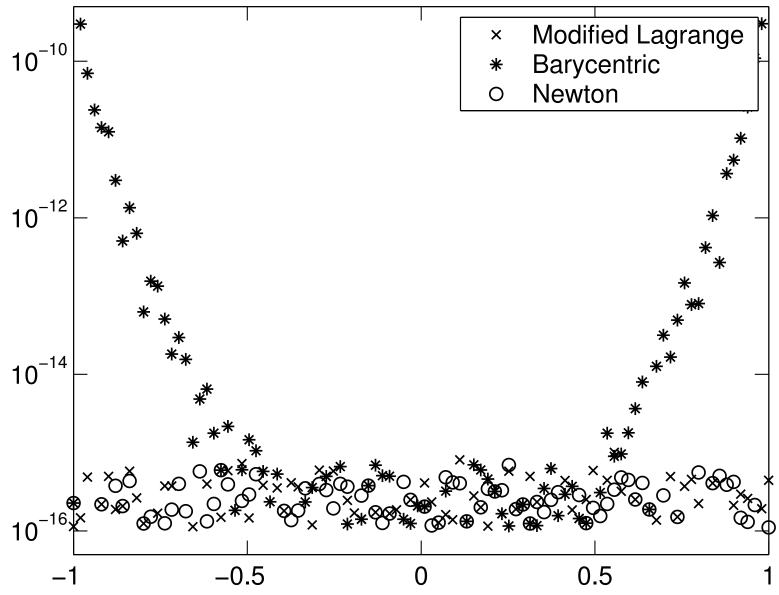

Figure 1: Relative errors in computed $p_n(x)$ for 30 equally spaced points x_i in increasing order.

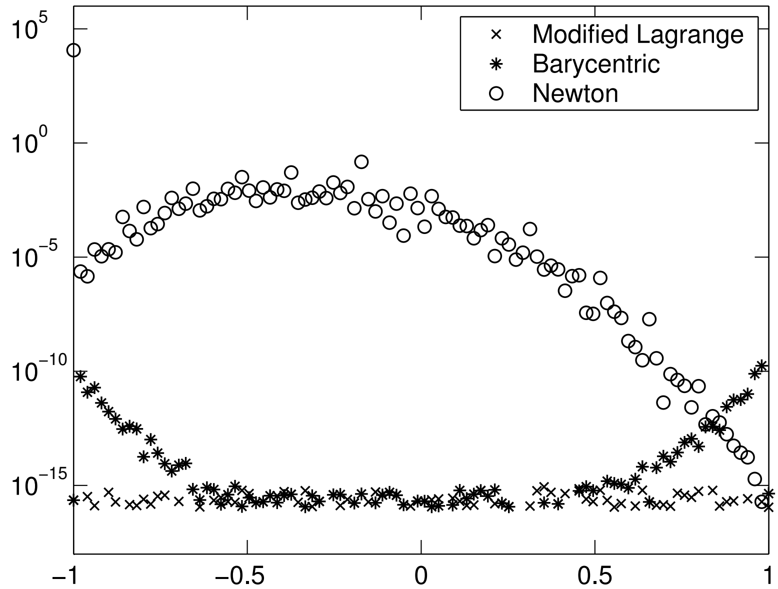

Figure 2: Relative errors in computed $p_n(x)$ for 30 equally spaced points x_i in decreasing order.

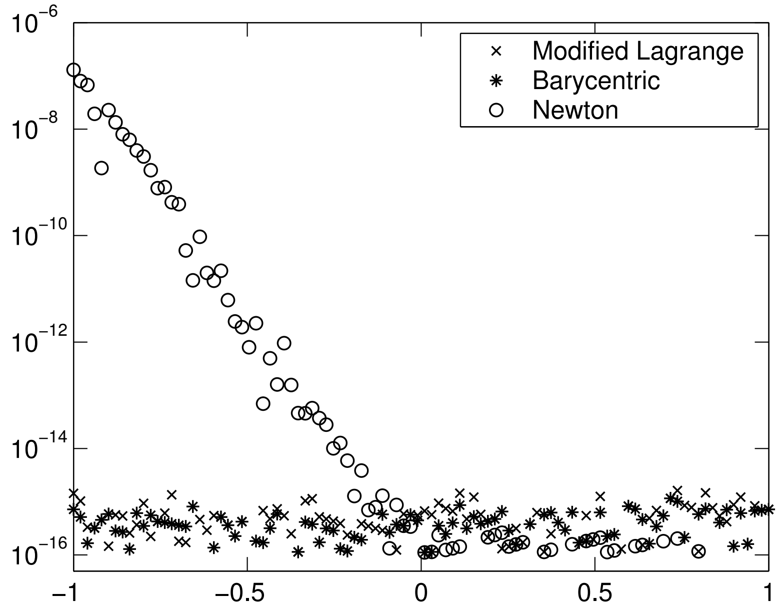

Figure 3: Relative errors in computed $p_n(x)$ for 30 Chebyshev points x_i of the first kind and $f(x) = 1/(1 + 25x^2)$.

the interval rather than equally spaced. Our analysis therefore provides support for the argument of Berrut and Trefethen [1] that barycentric Lagrange interpolation should be the interpolation method of choice.

We are not aware of any error analysis for construction and evaluation of the Newton divided difference formula at arbitrary points. The rounding error analysis in [6, Sec. 5.3], which covers construction followed by evaluation at the interpolation points only, could be extended to handle arbitrary points. However, it is clear from the analysis in [6, Sec. 5.3] and from the experiments reported here that the errors from the Newton form are very dependent on the ordering and can be unacceptably large even for Chebyshev points.

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