Purifying Natural Deduction Using Sequent Calculus

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Verified Programming

Thesis
The ability to state and prove properties of code is the crucial missing technology in the evolution of software.

- Stronger guarantees => less monitoring => higher performance.
- Ability to trust software opens up new applications.
- Confirmed quality helps open source, app stores, etc.
- Verification is a tool we don’t have.
The GURU Verified Programming Language (VPL)

Functional language
Dependently typed programs
General recursion
Notation for theorems, proofs about programs
Unaliased mutable state
Resource management layer
Type/Proof-checker, compiler to C
No concurrency
Aliasing for mutable state in progress
Basic GURU Design

- Terms : Types.
- Proofs : Formulas.
- “Full-spectrum” dependency.
  - Types can contain arbitrary terms (<list A \(n\)>).
  - Type checking decidable.
  - Explicit casts with proofs of \(\{T = T'\}\).
- Proofs and types can appear in terms.
  - computationally irrelevant.
  - erased by compilation, definitional equality.
The **Guru Compiler**

Guru source code

Parser

Type/proof-checker

Pull out λs

Resource analysis

Linearization

Compile datatypes

C target code

**CARRAWAY Layer**
Resource Management in GURU

- Resources: program data, I/O channels, mutable arrays.
- Resource typing side-by-side with data typing.
- Management policies definable.
- Based on fundamental idea of a resource:
  1. A resource can only be used by one entity at a time.
  2. A resource can be temporarily decomposed into subresources.
- Statically ensure all resources “consumed” exactly once.
Subresources

- “Goblet of Fire” as subresource of Harry Potter boxed set.
- Sublist \( l' \) as a subresource of \((\text{cons } x \ l')\).
- Subresource relationship based on type \(<R \ x>\):
  - \( x : R \) – \( x \) has resource type \( R \).
  - \( y : <R' \ x> \) – \( y \) has resource type \( R' \), and is a subresource of \( x \).
- Cannot consume \( x \) until all subresources have been consumed.
Example: Reference-Counted Data

- Guru uses reference counting for inductive data.
- Primitive \(\text{inc } x\) creates new view of \(x\).
- \(\text{dec } x\) consumes a view of \(x\).
- \textit{owned} resource type for loaned reference.

```plaintext
match l with            \% suppose \(l : <\text{owned } x>\)
    nil => ...            \% then \(l' : <\text{owned } l>\)
  \| \text{cons } x \ l' =>
```

- Must drop \(l'\) before consuming \(l\).
- Can increment \(l'\) to get new view.
- Sometimes must collapse chains of ownership:

```plaintext
\@ \ l' : <\text{owned } x>
```
Meta-Theoretic Concerns

- To implement a VPL: go from proof theory to compilers.
- “Practical” proof theory lacking.
- Problems with disjunctions ($\phi \lor \phi'$) and existentials ($\exists x. \phi$).
- Rest of the talk: the problems, and progress towards a solution.
How to prove your logic is consistent?

Basic strategy:

1. Identify subset of proofs which obviously are ok.
2. Define rewrite rules to transform any proof to one in the ok form.
3. Prove rules are (strongly or weakly) normalizing.

By Curry-Howard isomorphism:

- Proofs are $\lambda$-terms.
- Proof normalization is $\beta$-reduction.

Reducibility proofs are powerful, elegant.

But do not work well with disjunctions, existentials.
Reducibility for Conjunction

Proof terms $p ::= (p_1, p_2) \mid p.1 \mid p.2$

\[
\begin{align*}
\Gamma \vdash p_1 : \phi_1 & \quad \Gamma \vdash p_2 : \phi_2 \\
\hline
\Gamma \vdash (p_1, p_2) : \phi_1 \land \phi_2
\end{align*}
\]

\[
\begin{align*}
\Gamma \vdash p : \phi_1 \land \phi_2 & \quad i \in \{1, 2\} \\
\hline
\Gamma \vdash p.i : \phi_i
\end{align*}
\]

Reducibility is “hereditary normalization”, defined by eliminations.

- $\text{Red}_\phi$ is set of reducible terms of type $\phi$.
- $p \in \text{Red}_b \iff \text{SN}(p)$, for base types $b$.
- $p \in \text{Red}_{\phi_1 \land \phi_2} \iff p.1 \in \text{Red}_{\phi_1}$ and $p.2 \in \text{Red}_{\phi_2}$.
- $p \in \text{Red}_{\phi_1 \rightarrow \phi_2} \iff \text{forall } p' \in \text{Red}_{\phi_1}, (p \; p') \in \text{Red}_{\phi_2}$
What Goes Wrong with Disjunction

Proof terms $p ::= \langle 1, p \rangle \mid \langle 2, p \rangle \mid \text{case}(p)(x.p_1, x.p_2)$

\[
\Gamma \vdash p : \phi_i \quad i \in \{1, 2\} \\
\frac{}{\Gamma \vdash \langle i, p \rangle : \phi_1 \land \phi_2} \quad \lor I
\]

\[
\frac{\Gamma \vdash p : \phi_1 \lor \phi_2 \quad \Gamma, x : \phi_1 \vdash p_1 : \psi \quad \Gamma, x : \phi_2 \vdash p_2 : \psi}{\Gamma \vdash \text{case}(p)(x.p_1, x.p_2) : \psi} \quad \lor E
\]

Attempt to define reducibility fails:

\[
p \in \text{Red}_{\phi_1 \lor \phi_2} \iff \forall \psi, \ p_1, p_2 \in \text{Red}_{\psi}, \ \text{case}(p)(x.p_1, x.p_2) \in \text{Red}_{\psi}
\]

Not legal to appeal to $\text{Red}_{\psi}$.
A Way Forward

- Problem with $\lor\ E$:
  - to use $p : \phi$, need $p' : \psi$, where $\psi$ unrelated to $\phi$.
  - breaks definition of reducibility.

- But compare sequent calculus rules:

  $$
  \frac{\Gamma, \phi_1 \vdash \psi \quad \Gamma, \phi_2 \vdash \psi}{\Gamma, \phi_1 \lor \phi_2 \vdash \psi} \quad \text{L}_\lor
  \quad
  \frac{\Gamma, \phi_1, \phi_2 \vdash \psi}{\Gamma, \phi_1 \land \phi_2 \vdash \psi} \quad \text{L}_\land
  $$

- Term assignment for sequent calculus is strange.

  $$
  \frac{\Gamma, y : \phi_1, z : \phi_2 \vdash p : \psi}{\Gamma, x : \phi_1 \land \phi_2 \vdash [x.1/y, x.2/z]p : \psi} \quad \text{L}_\land
  $$

- Limited by old view of “natural” deduction.
A Direct Term Assignment

- Left rules correspond to eliminations.
- Why insist that the context $\Gamma$ holds just variables?
- Proposal:
  - Assign terms to sequent calculus directly.
  - Devise new terms for $\vee E$, $\exists E$.
  - Allow $\Gamma$ to hold terms.
### Elimination Rules

<table>
<thead>
<tr>
<th>Rule</th>
<th>Description</th>
</tr>
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<tbody>
<tr>
<td><strong>L^\wedge</strong></td>
<td>(\Gamma, p.1 : \phi_1, p.2 : \phi_2 \vdash p' : \psi) [\Gamma, p : \phi_1 \land \phi_2 \vdash p' : \psi]</td>
</tr>
<tr>
<td><strong>L^\vee</strong></td>
<td>(\Gamma, p.1 : \phi_1 \vdash p_1 : \psi) [\Gamma, p.2 : \phi_2 \vdash p_2 : \psi] [\Gamma, p : \phi_1 \lor \phi_2 \vdash p_1 \lor p_2 : \psi]</td>
</tr>
<tr>
<td><strong>L^\forall</strong></td>
<td>(\Gamma, (p \ a) : [a/x]\phi \vdash p' : \psi) [\Gamma, p : \forall x.\phi \vdash p' : \psi]</td>
</tr>
<tr>
<td><strong>L^\exists</strong></td>
<td>(\Gamma, p !x : \phi \vdash p' : \psi) [x \not\in \text{FV}(\Gamma, \psi)] [\Gamma, p : \exists x.\phi \vdash \nu x.p' : \psi]</td>
</tr>
<tr>
<td><strong>L^\rightarrow</strong></td>
<td>(\Gamma, p_2 : \phi_2 \Gamma, (p_1 \ p_2) : \phi_1 \vdash p' : \psi) [\Gamma, p_1 : \phi_2 \rightarrow \phi_1 \vdash p' : \psi]</td>
</tr>
<tr>
<td><strong>L^W</strong></td>
<td>(\Gamma, p : \phi \vdash [p]p' : \psi)</td>
</tr>
<tr>
<td><strong>L^C</strong></td>
<td>(\Gamma, p : \phi \vdash p' : \psi)</td>
</tr>
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</table>
Reduction

- We have separated logical terms \((t.(i))\) from structural \((t_1 \parallel t_2)\).
- Logical terms have \(\beta\)-reductions:
  \[
  (t_1, t_2).i \rightsquigarrow t.i \\
  \langle i, t \rangle.(i) \rightsquigarrow t \\
  \langle i, t \rangle.(3 - i) \rightsquigarrow \text{abort}
  \]
- Structural terms have commuting conversions:
  \[
  (t_1 \parallel t_2).i \rightsquigarrow (t_1.i) \parallel (t_2.i) \\
  \text{abort} \parallel t \rightsquigarrow t
  \]
- Simple unsound typing rules suffice for reducibility.
  \[
  \Gamma \vdash p : \phi_1 \lor \phi_2 \\
  \Gamma \vdash p.i : \phi_i \quad \lor E
  \]
Towards Pure Natural Deduction

- Next step: define sound natural deduction rules.

\[ J ::= \Gamma \vdash \Delta \mid J \parallel J \]
\[ \Delta ::= \ t_1 : \phi_1, \ldots, t_n : \phi_n \]

- Prove type preservation.
- Prove confluence.
- Final result: Pure Natural Deduction.
  - All rules are either direct logical rules or structural.
  - Consistency proved by reducibility.
  - Decidable equational theory, including commuting conversions.
  - Practical proof theory ready to use for VPL.

www.guru-lang.org