A Counterexample Based Approach for Quantifier Instantiation in SMT

Andrew Reynolds, University of Iowa
MVD, September 30, 2011

Overview

- Introduction to Satisfiability Modulo Theories (SMT)
- Extending SMT to Quantifiers
- Approaches to Quantifier Instantiation
  - E-Matching
  - Model-Based Quantifier Instantiation
  - New: Counterexample-Based Approach
- Current Work

Satisifiability Modulo Theories (SMT)

- SMT extends boolean satisifiability problems to *theories*

$$F = \{ (f(c) = a \vee c + 4 > a), (a = g(b)) \}$$

- Construct satisfying assignment M for set of clauses F
  - i.e. $M = \{ f(c) = a, a = g(b) \}$
- Is this assignment consistent according to theory reasoning?

DPLL(T) Architecture

- SMT uses DPLL(T) architecture
- Operates on states of the form

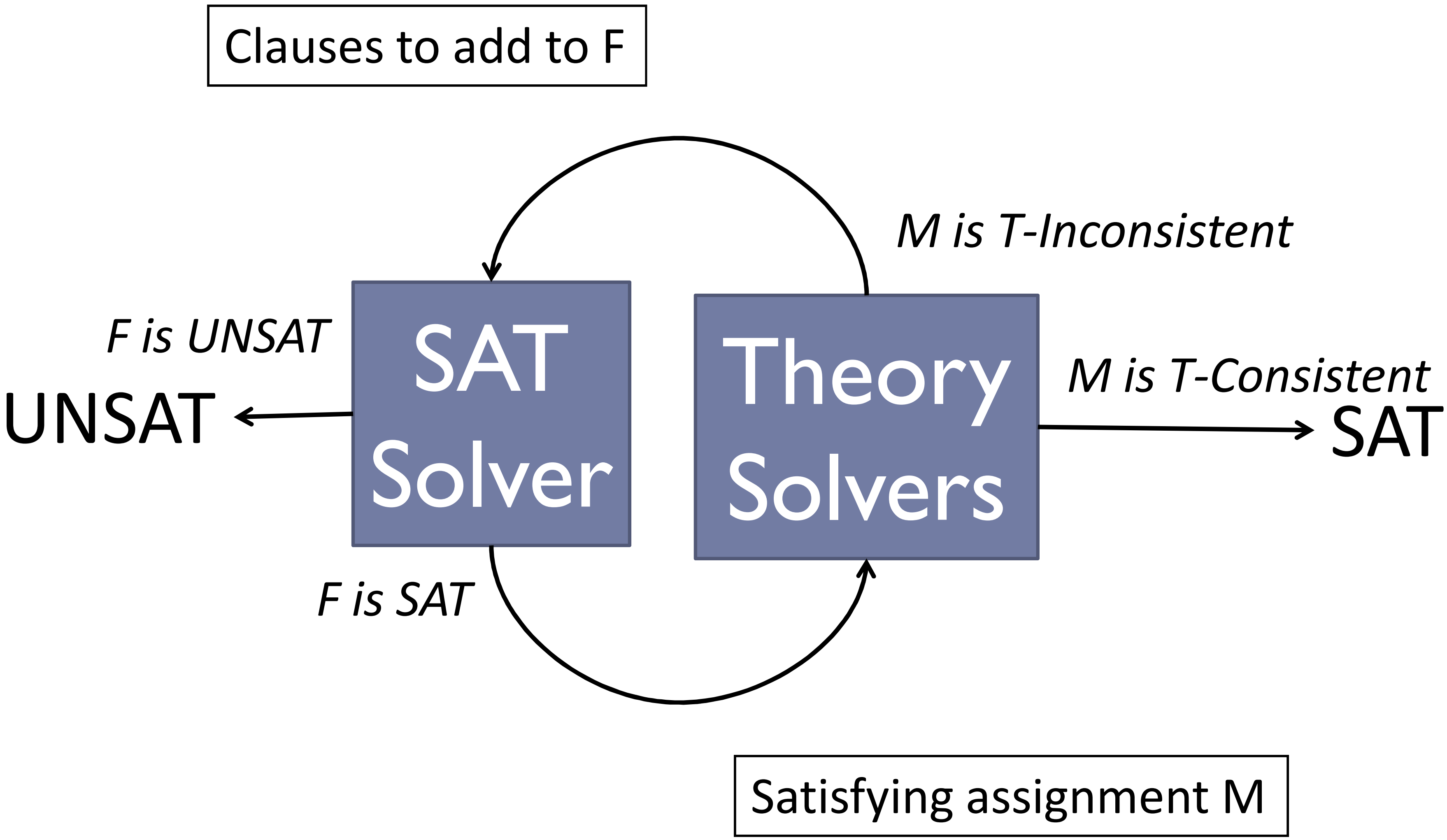
$$M \parallel F$$

- F is a set of clauses
- M is a set of asserted theory literals "L"
  - Literals may be decisions "L^d"

DPLL(T) Architecture

- For a DPLL(T) state M ‖ F,
  - SMT solver can answer UNSAT if:
    - Some clause in F is falsified by M, and
    - M contains no decision literals L^d
  - SMT solver can answer SAT if:
    - Each clause in F is satisfied in M, and
    - M is T-consistent

DPLL(T) Architecture



Role of Theory Solver T in SMT

- Accepts a set of theory literals M
  - Determine if M is T-consistent
  - If not, add *lemmas* C to F, where each C is T-valid
- Typically, use SMT for decidable logics
  - Quantifier-free UF, Linear Real Arithmetic, etc.
- Also may be interested in other logics
  - Non-linear arithmetic, quantified logics, etc.

Quantifiers in SMT

- Universal and existential quantifiers
  - $\forall x.\ \phi$, $\exists x.\ \phi$
  - Treated as literals by the SAT solver
- Relegate these literals to quantifiers module
  - Role is similar to theory solver
  - Checking T-consistency is undecidable
    - When $\forall x.\ \phi$ is asserted, cannot answer SAT
- When asked whether M is T-consistent, and there is a $\forall x.\ \phi$ asserted in M, either:
  - Answer UNKNOWN
  - Add (instantiation) clause ($\neg\forall x.\phi \vee \phi[s/x]$) to M

Quantifiers in SMT: Challenges

(1) Finding relevant instantiations

- How do we determine ground term s?

(2) Deciding when providing instantiations is no longer worthwhile

- When should we answer UNKNOWN?

(3) Determining if all necessary instantiations have been applied

- Can we answer SAT?

Related Work: E-matching

- Address challenge (1)
  - Find relevant instantiations by matching terms in quantifiers t[x] to ground terms t[s/x]
- To construct instantiation for $\forall x.\phi$:
  - Find *trigger* t, where x is in FV(t)
  - Find ground term g
  - Find substitution [s/x] such that t[s/x] is equivalent to g modulo set of equalities E
    - *"t E-matches g"*
  - Use s to instantiate $\forall x.\phi$

Related Work: Model-Based Quantifier Instantiation (MBQI)

- Address challenges (1) and (3)
  - Determine if some model satisfies all quantifiers. If so, answer SAT. Otherwise, use values for which model fails to instantiate quantifiers.
- Given asserted quantified formula $\forall x.\phi$:
  - Build explicit model M^I for ground clauses F
  - Replace uninterpretted symbols in ϕ to generate ϕ^I
  - Determine the satisfiability of $R \wedge \neg\phi^I[e/x]$
  - If UNSAT, then $\forall x.\phi$ is valid in current context
    - Otherwise, model for $R \wedge \neg\phi^I[e/x]$ is used to instantiate $\forall x.\phi$
    - Rules out M^I on subsequent iterations

MBQI Example

- Check satisfiability of $F \wedge \phi$

  F: $w \geq v + 2 \wedge f(v) \leq 1 \wedge f(w) \leq 3$

  ϕ: $\forall\, i\, j.\, (i \leq j \Rightarrow f(i) \leq f(j))$

- Model M^I for F:

  $v \rightarrow 0, w \rightarrow 2, f \rightarrow [\, 0 \rightarrow 1, 2 \rightarrow 3, \text{else} \rightarrow 4 \,]$

- Check satisfiability of $\neg\phi^I[e_i/i, e_j/j]$:

  $e_i \leq e_j \wedge \text{ite}(e_i{=}0, 1, \text{ite}(e_i{=}2, 3, 4)) = \text{ite}(e_j{=}0, 1, \text{ite}(e_j{=}2, 3, 4))$

Alternative Approach to MBQI

- MBQI builds explicit models M^I
  - Check sat for $R \wedge \neg\phi^I[e/x]$
- *Instead:* Reason about counterexample e directly
  - Add clause containing $\neg\phi[e/x]$ to SMT solver
- Potential advantages:
  - Do not need to generate explicit models M^I
  - Reason about $\neg\phi[e/x]$ incrementally, using the same instance of SMT solver

Counterexample Lemma

- Write $\perp^{\phi}$ to denote literal meaning:

  "a counterexample to ϕ exists"

- SMT solver finds satisfying assignment to:

$$(\phi \vee \perp^{\phi})$$

"either ϕ holds or a ϕ has a counterexample"

$$(\perp^{\phi} \Leftrightarrow \neg\phi[e/x])$$

"ϕ has a counterexample if and only if its negation holds for some value e"

Configurations for Quantifier/CE Literal

- ϕ is not asserted in M
  - We don't care about ϕ
- $\phi^{(d)}$ and $(\perp^{\phi})^{d}$ are asserted in M
  - ϕ is true but we might find a counterexample
- $\phi^{(d)}$ and $\neg\perp^{\phi}$ are asserted in M
  - ϕ is true and we know it does not have a counterexample

- *Requirement: Never assert* $\neg(\perp^{\phi})^{d}$

Recognizing SAT Instances with CE Literals

- If $\bot^{\phi}$ is asserted negatively *as a non-decision*, then ϕ is valid in the current context
  - If this is true for all quantifiers ϕ, then we may answer SAT
- Conceptually: axiom ϕ does not apply in the current context

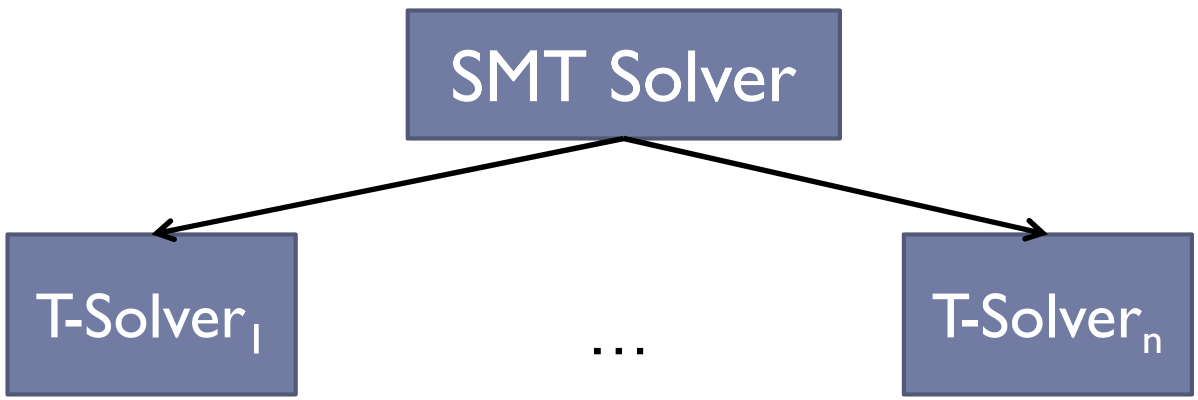
- Example: $a=0 \wedge (\forall\ x.\ a > 0 \Rightarrow P(a, x))$
  - $\bot^{\phi} \Leftrightarrow (a > 0 \wedge \neg P(a, e))$

Features of Counterexample-Based Approach

- May be able to recognize SAT instances
  - Cases when no quantified axiom applies, i.e. counterexample is unsatisifiable
- Use information about "e" for finding relevant instantiations
  - Theory-specific information

Theory-Specific Instantiators

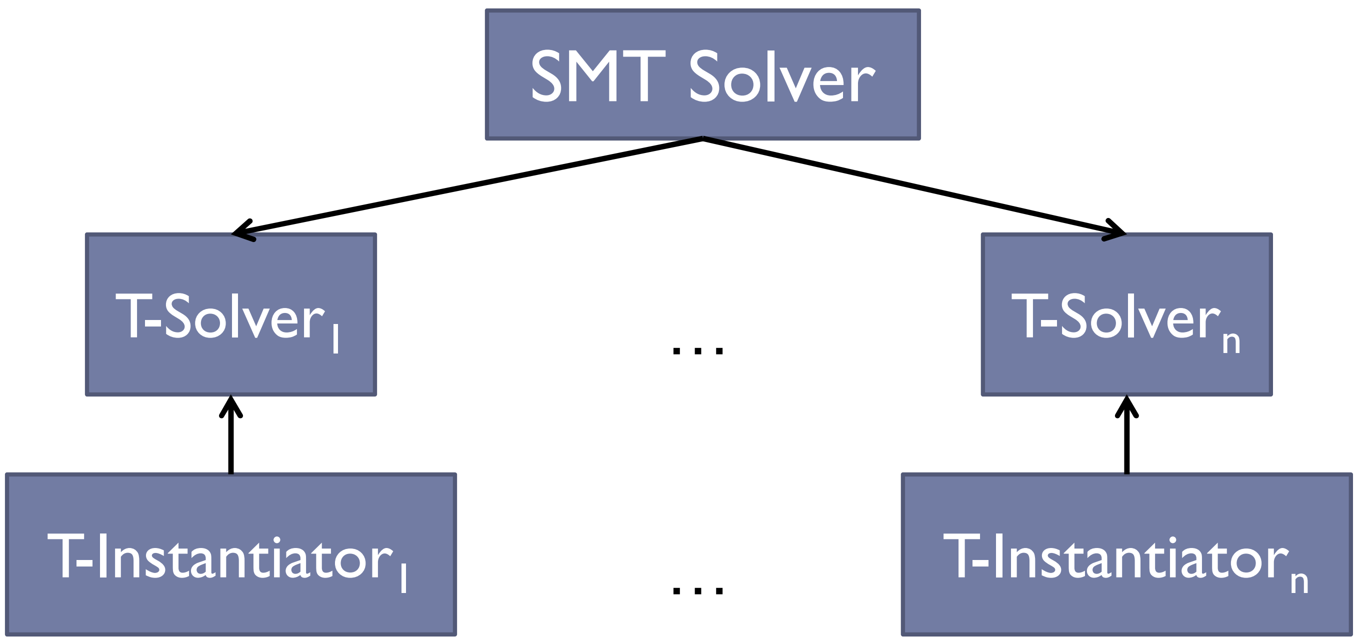
- After finding satisfying assignment to $\neg\phi[e/x]$
  - Each theory solver has theory-specify information/constraints involving e



- How can we use this information?
  - Naively, find arbitrary model and use value of e to instantiate ϕ

Theory-Specific Instantiators

- Can we do better?
- For each theory, associate an *instantiatior*
  - Has access to internal information stored in theory solver



Using Relationships between Triggers

- For EUF:
  - Search method for finding relevant instantiations
    - For literal t[e/x] = s, first try to find match t[g/x] *in the equivalence class of s*
  - Criteria for judging relevance of instantiations
    - Do not consider instantiations g where e = g is unsatisifiable

Quantifier Instantiation for EUF

- Multiple Iterations:
  1. Find if e = s is entailed for some ground term s
  2. Find if there exists some s such that all requirements for e are entailed by e = s
  3. Find if there exists some s such that some requirements for e are (partially) matched by e = s
  4. Do E-matching

- Otherwise, see if (explicit) model can be constructed

Current Work

- Optimizations
  - Computing matches efficiently (i.e. indexing, caching)
- Using splitting on demand
  - Matching failed because c_1 and c_2 are not entailed to be equal
  - Add lemma ($c_1 = c_2 \vee c_1 \neq c_2$)
- Quantifier Instantiation for Arithmetic
- Recognizing Other SAT instances
  - If no matches can be found, construct explicit model M^I and see if MBQI succeeds
  - Construction of M^I based on information about e
- Backtracking decisions
  - If stuck, explore another part of the search space

Questions?