Induction in CVC4

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Overview

- Satisfiability Modulo Theories (SMT)
  - SMT solver CVC4
- Induction techniques in CVC4 [Reynolds/Kuncak, 2015]
  - Inductive strengthening
  - Subgoal generation
  - Leveraging theory reasoning
- Experiments

Automated Reasoning

- Historically, automated reasoning meant uniform proof procedures for FOL
- More recent trend is decidable fragments
  - Equality
  - Arithmetic
  - Data structures (arrays, lists, records)
  - …

Automated Reasoning

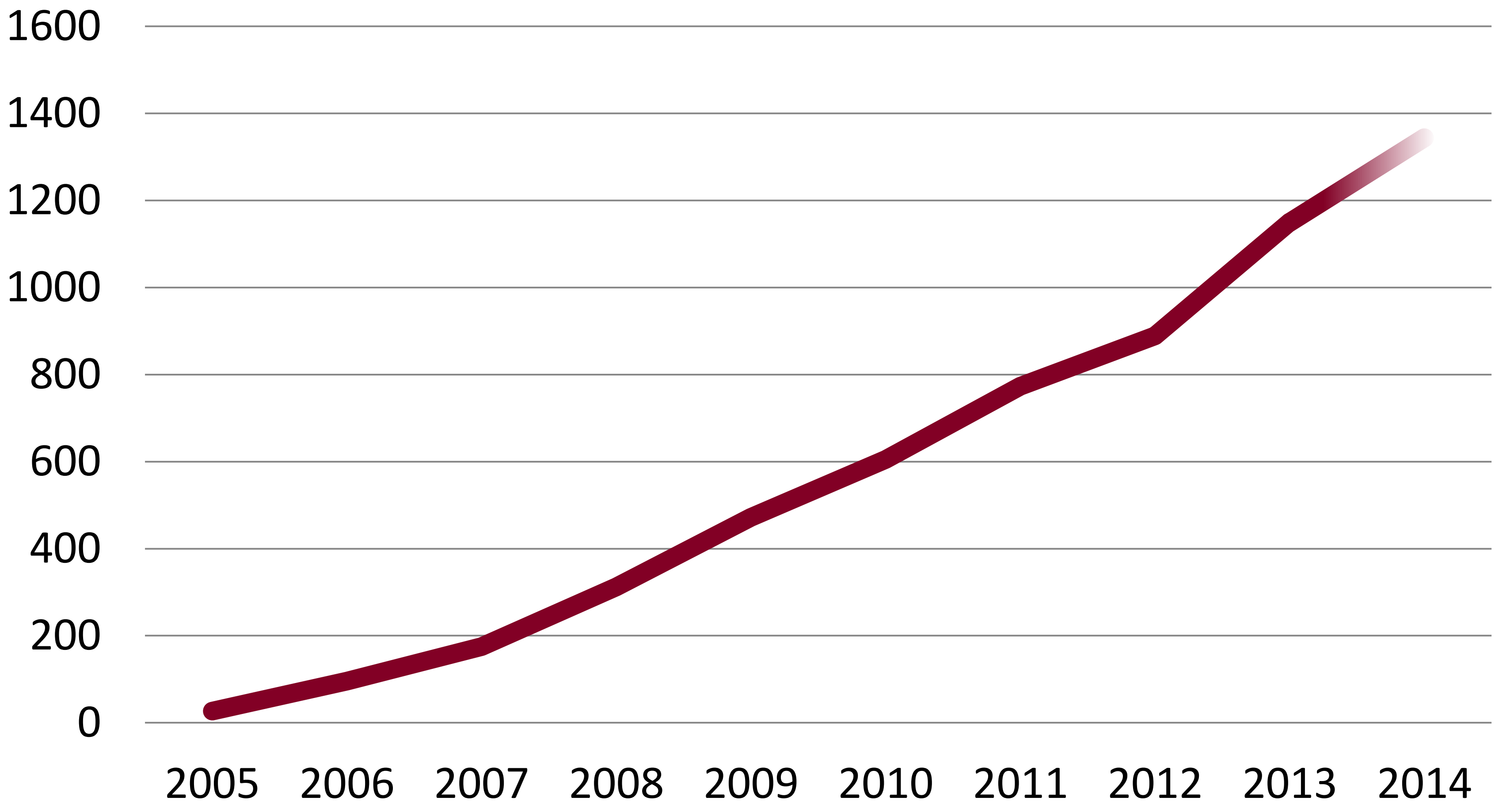
- Examples
  - SAT – propositional, Boolean reasoning
    - Efficient
    - Expressive (NP) but involved encodings

$$(A \vee \neg B) \wedge (\neg A \vee C \vee D)$$

  - SMT – first order, Boolean + DS reasoning
    - Loss of efficiency
    - Improves expressivity and scalability

$$(x+1>0 \vee \neg a[x]=b) \wedge (\neg P(y) \vee y=z)$$

Articles mentioning SMT over time



Applications of SMT

- Extended static checking
- Predicate abstraction
- Model checking
- Scheduling
- Test generation
- Synthesis
- Verification

CVC4

- State-of-the-art SMT Solver
  - Developed over last 5 years as successor of CVC3
- Supports many theories:
  - Arithmetic, Arrays, Bitvectors, UF
  - Inductive/Co-inductive Datatypes
  - New : Strings, Floating Points, Finite Sets
- Has strong performance:
  - Placed 1st in 14 of 32 divisions of SMT-COMP
  - Won TFA division of CASC J7
  - Competitive for many common SMT uses

The CVC4 Team


Clark Barrett (NYU)
Cesare Tinelli (U Iowa)



Kshitij Bansal (NYU)
François Bobot (CEA)
Chris Conway (Google)



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Liana Hadarean (NYU)
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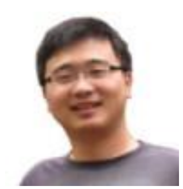



Tim King (Verimag)
Tianyi Liang (U Iowa)
Andrew Reynolds (EPFL)

CVC4 : Quantifiers

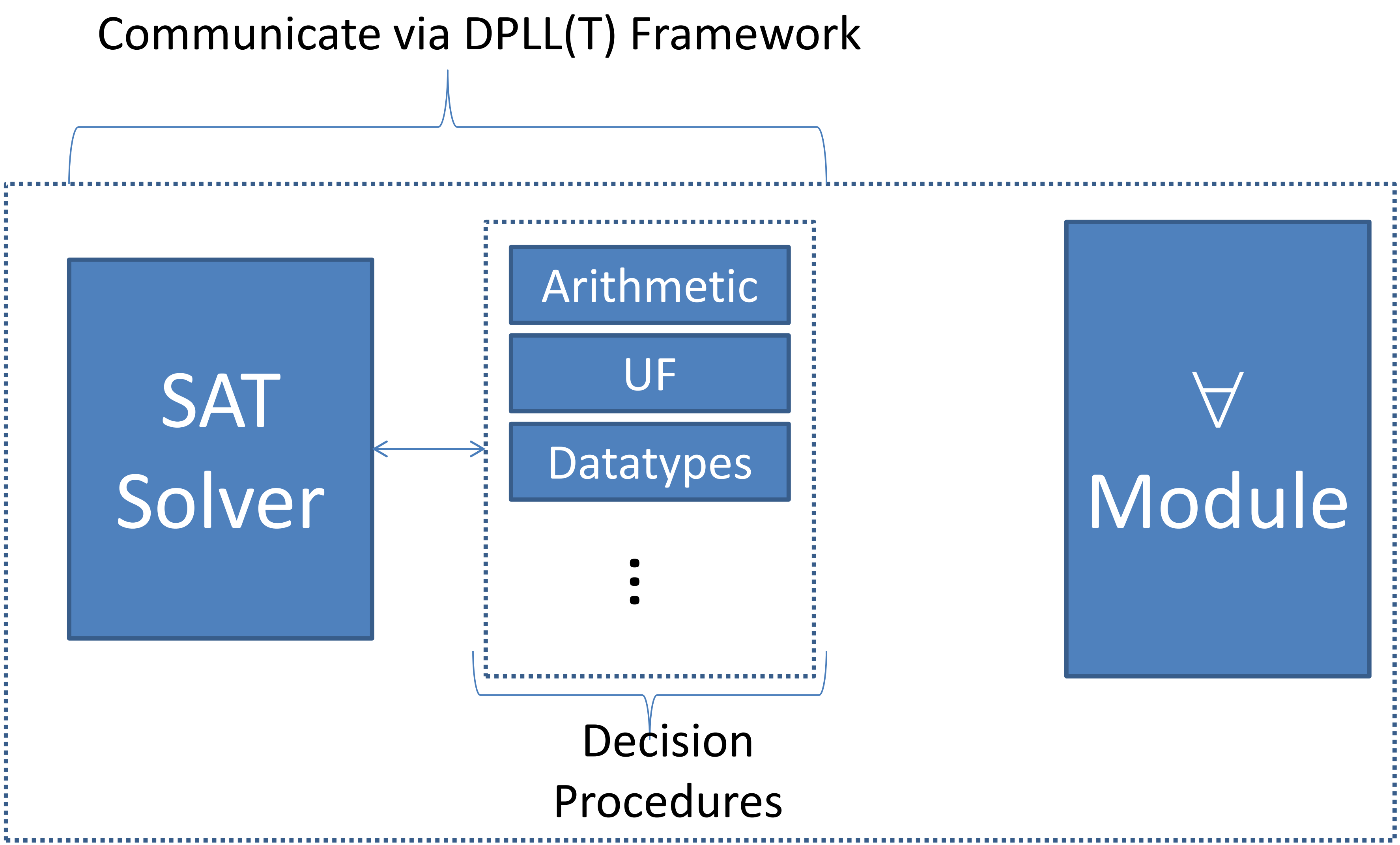
- Handles (universally) quantified formulas

$$\underbrace{\forall x : T.}_{\text{for all x of type T}} P(\, x \,)$$

$\Rightarrow$Satisfiability problem with $\forall$ is generally undecidable

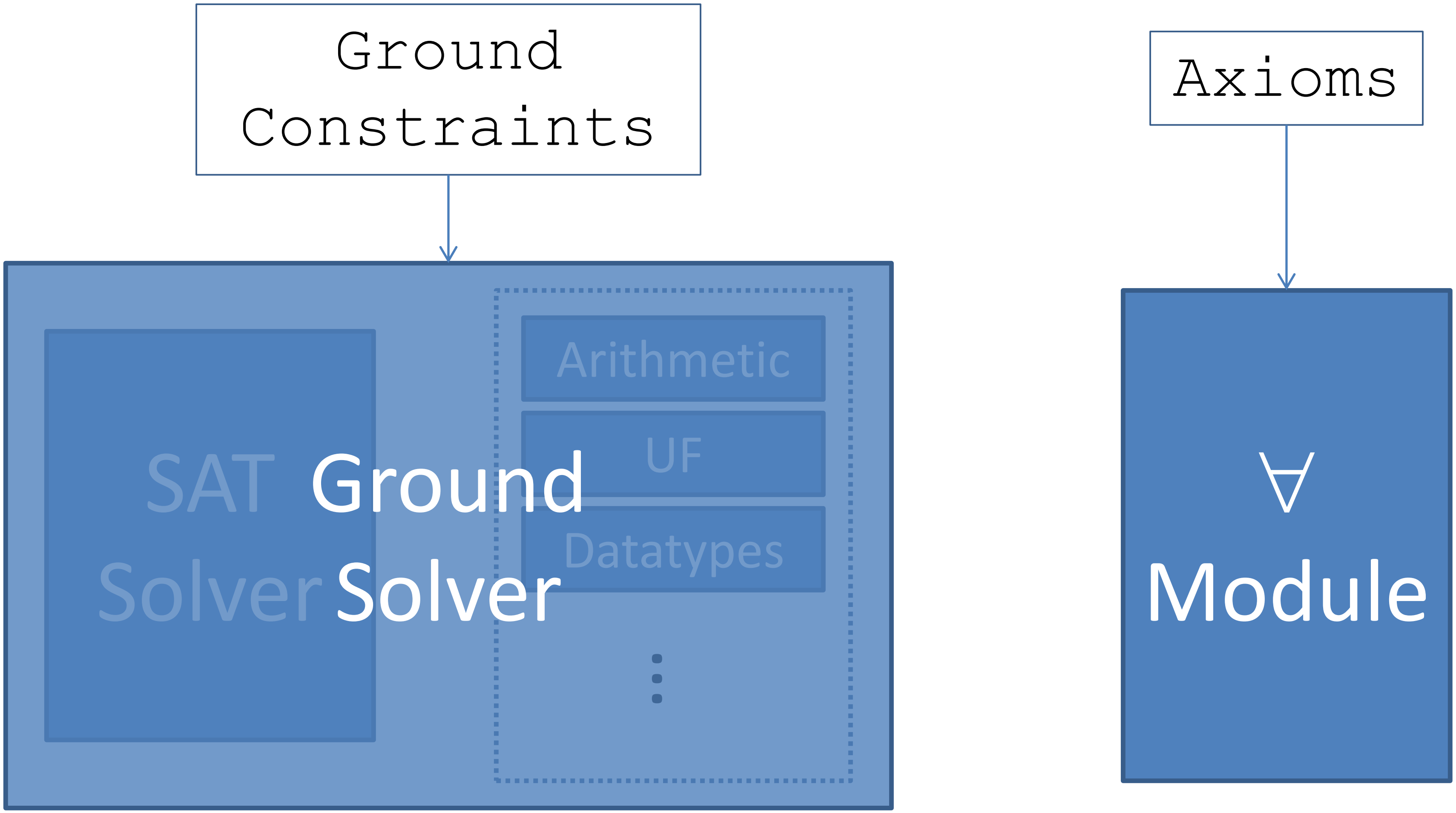
- CVC4 handles quantifiers by:
  - Heuristic instantiation (E-matching)
  - Conflict-based instantiation [FMCAD 2014]
  - Finite model finding/model-based instantiation [CADE/CAV 2013]
  - Rewrite Rules
  - New: Syntax-Guided Synthesis, Induction [VMCAI 2015]

SMT Solver

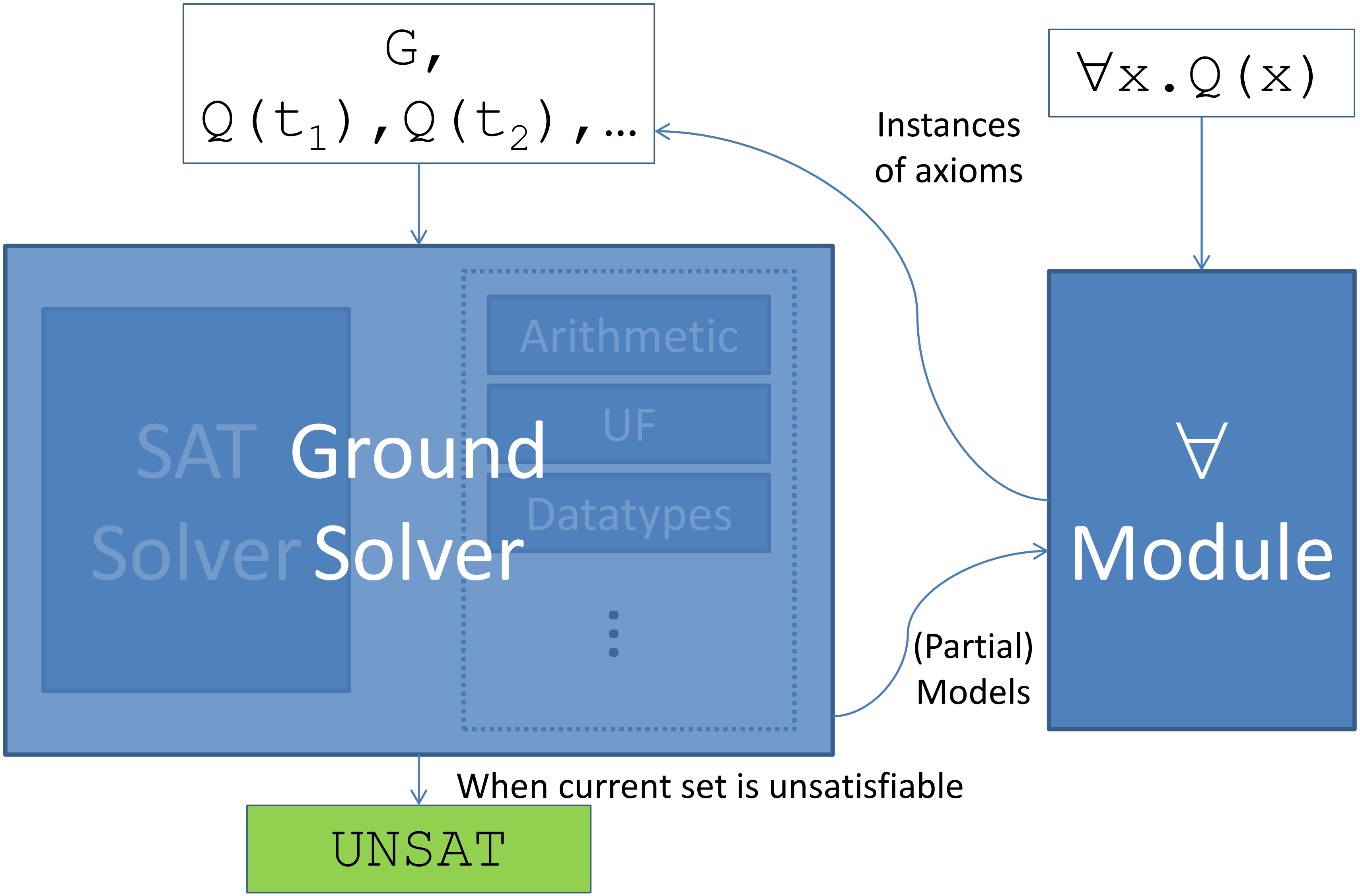


Communicate via DPLL(T) Framework
SAT Solver
Arithmetic
UF
Datatypes
⋮
∀ Module
Decision Procedures

SMT Solver


Ground Constraints
Axioms
SAT Solver
Ground Solver
Arithmetic
UF
Datatypes
∀ Module

SMT Solver


G,
Q(t1),Q(t2),…
∀x.Q(x)
Instances
of axioms
Arithmetic
UF
Datatypes
SAT Solver
Ground Solver
∀
Module
(Partial)
Models
When current set is unsatisfiable
UNSAT

Running Example

- Datatype `List`

```
List:= cons(hd:Int,tl:List) | nil
```

- Length function `len : List -> Int`

```
len(nil)=0,
∀xy.len(cons(x,y))=1+len(y)
```

Example #1 : Ground Conjecture

```
len(nil)=0
∀xy.len(cons(x,y))=1+len(y)
```

Axioms

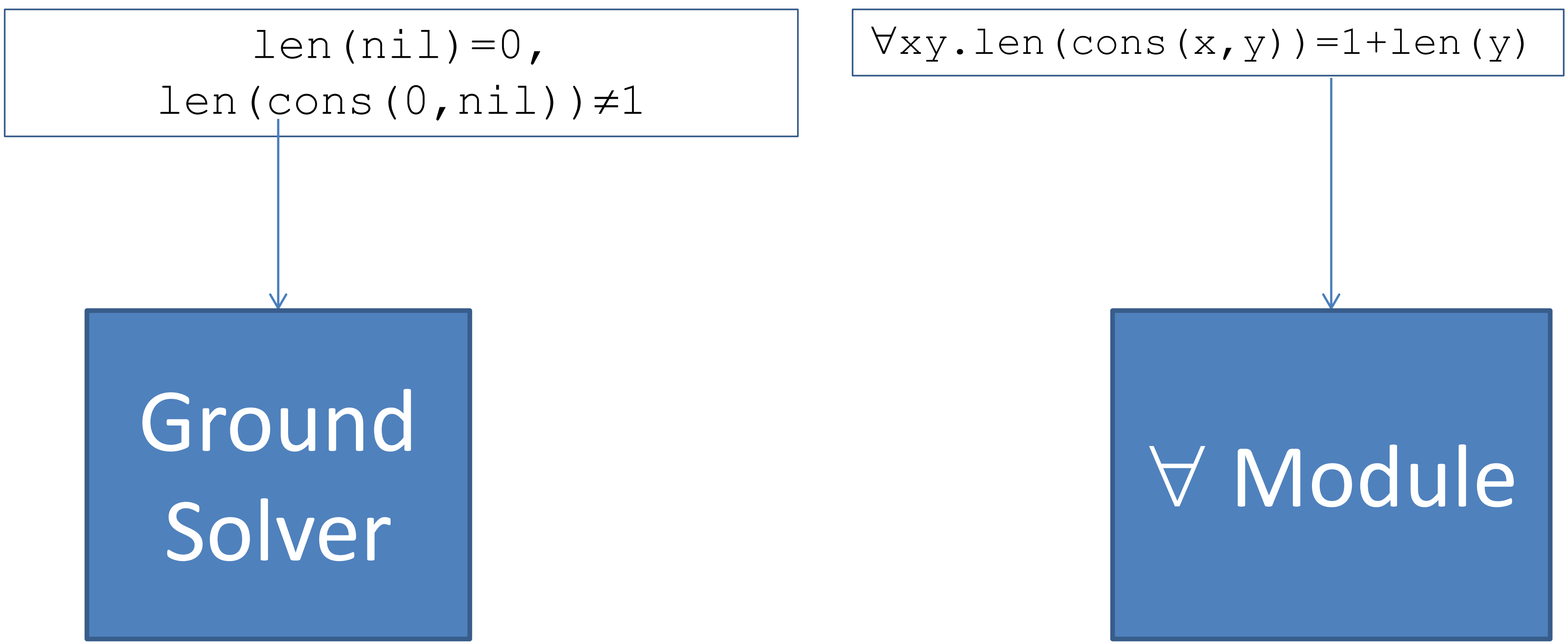
```
¬len(cons(0,nil))=1
```

(Negated) Conjecture

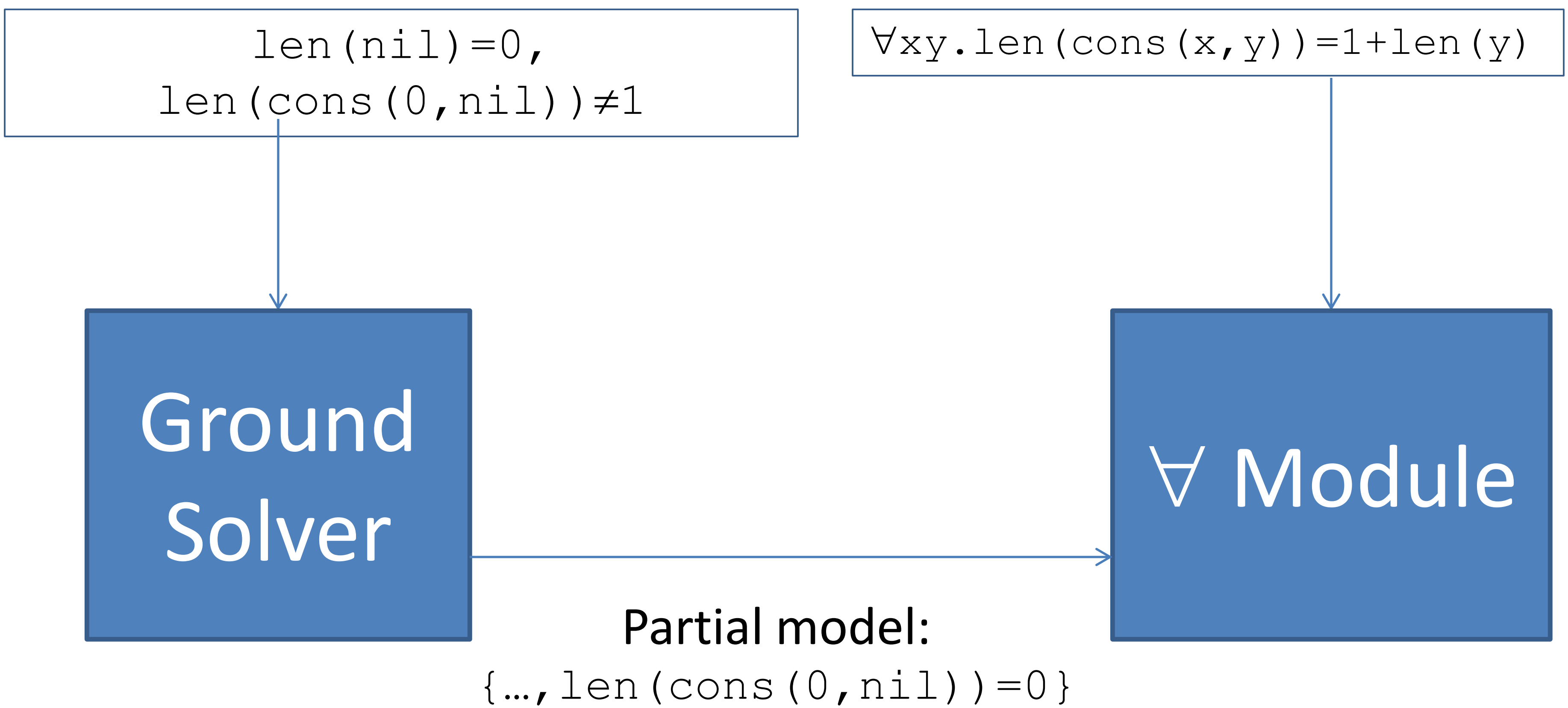
Ground Solver

∀ Module

Example #1

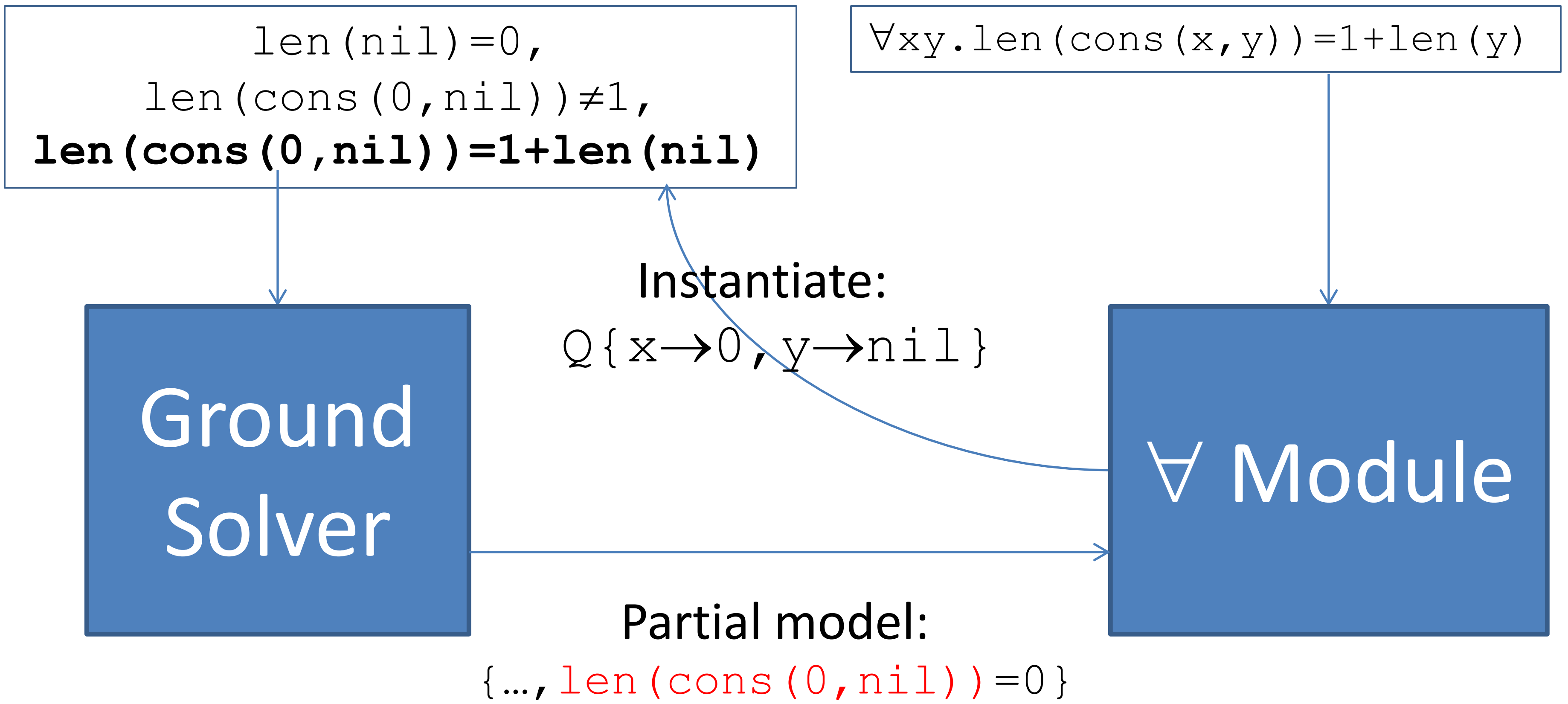

len(nil)=0,
len(cons(0,nil))≠1
∀xy.len(cons(x,y))=1+len(y)
Ground Solver
∀ Module

Example #1



len(nil)=0,
len(cons(0,nil))≠1
∀xy.len(cons(x,y))=1+len(y)
Ground Solver
∀ Module
Partial model:
{…,len(cons(0,nil))=0}

Example #1



len(nil)=0,
len(cons(0,nil))≠1,
len(cons(0,nil))=1+len(nil)
∀xy.len(cons(x,y))=1+len(y)
Instantiate:
Q{x→0,y→nil}
Ground Solver
∀ Module
Partial model:
{…,len(cons(0,nil))=0}

Example #1

`len(nil)=0,`
`len(cons(0,nil))≠1,`
`len(cons(0,nil))=1+len(nil)`

`∀xy.len(cons(x,y))=1+len(y)`

Ground Solver

∀ Module

`UNSAT`

Since `len(cons(0,nil))=1+len(nil)=1+0=1≠1`

Example #2 : Quantified Conjecture

```
len(nil)=0
∀xy.len(cons(x,y))=1+len(y)
```

Axioms

```
¬∀x.len(x)≥0
```

(Negated) Conjecture

Ground Solver

∀ Module

Example #2

```
len(nil)=0
∀xy.len(cons(x,y))=1+len(y)
```

```
¬∀x.len(x)≥0
```

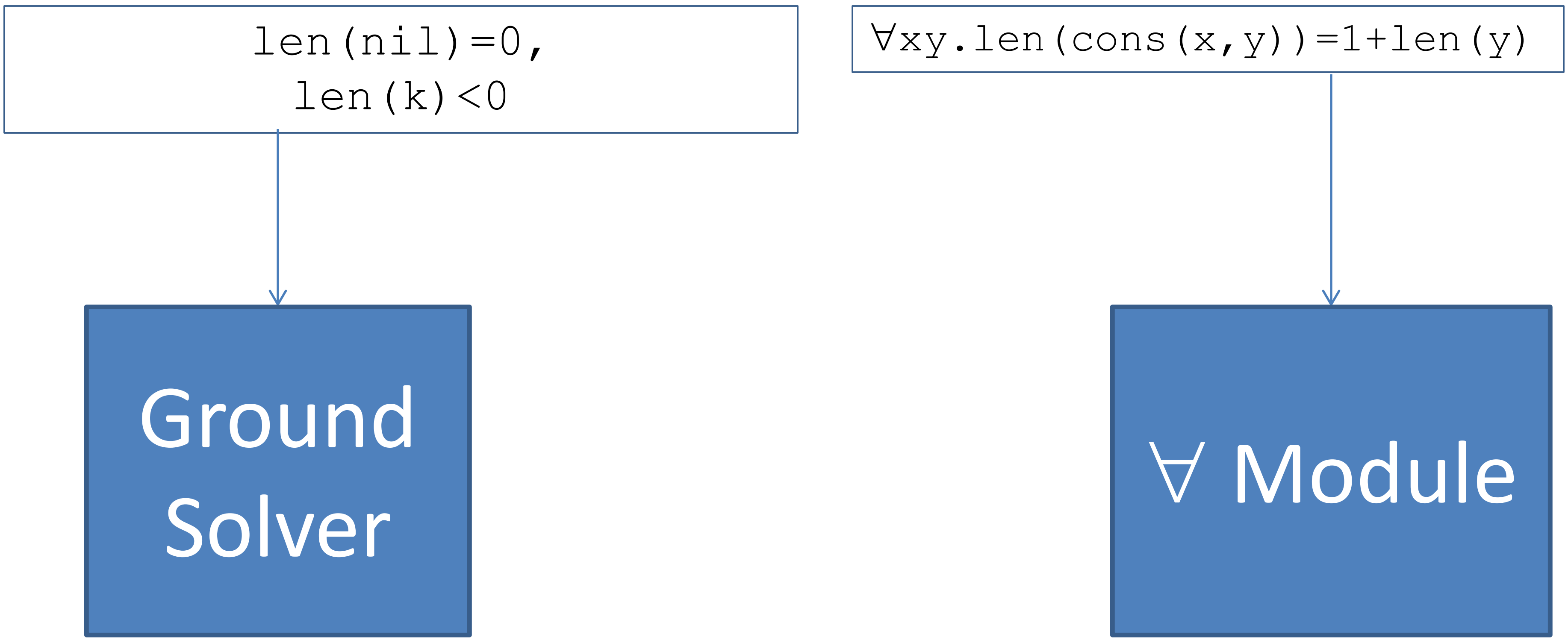
Skolemize : statement (does not) hold for fresh constant **k**

```
¬  len(k)≥0
```

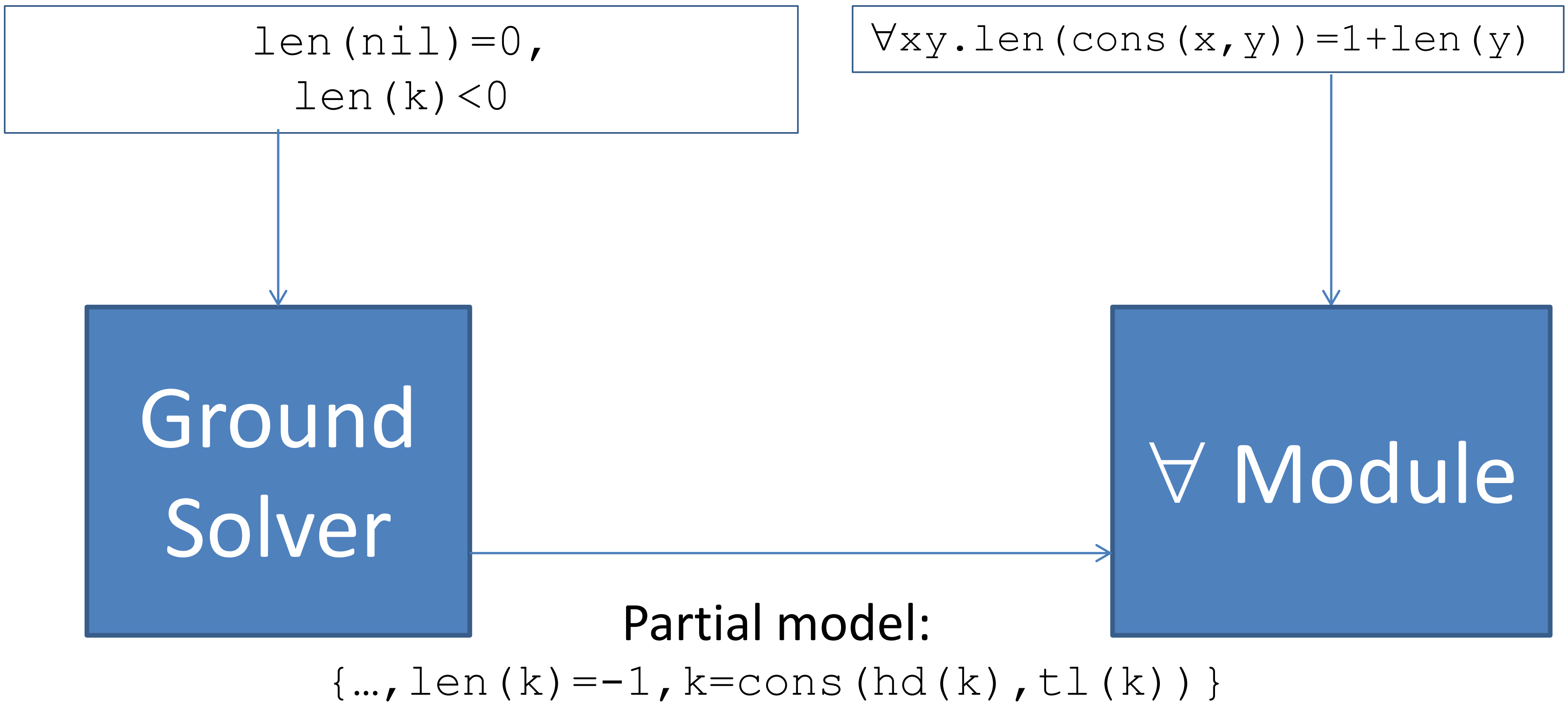
Ground Solver

∀ Module

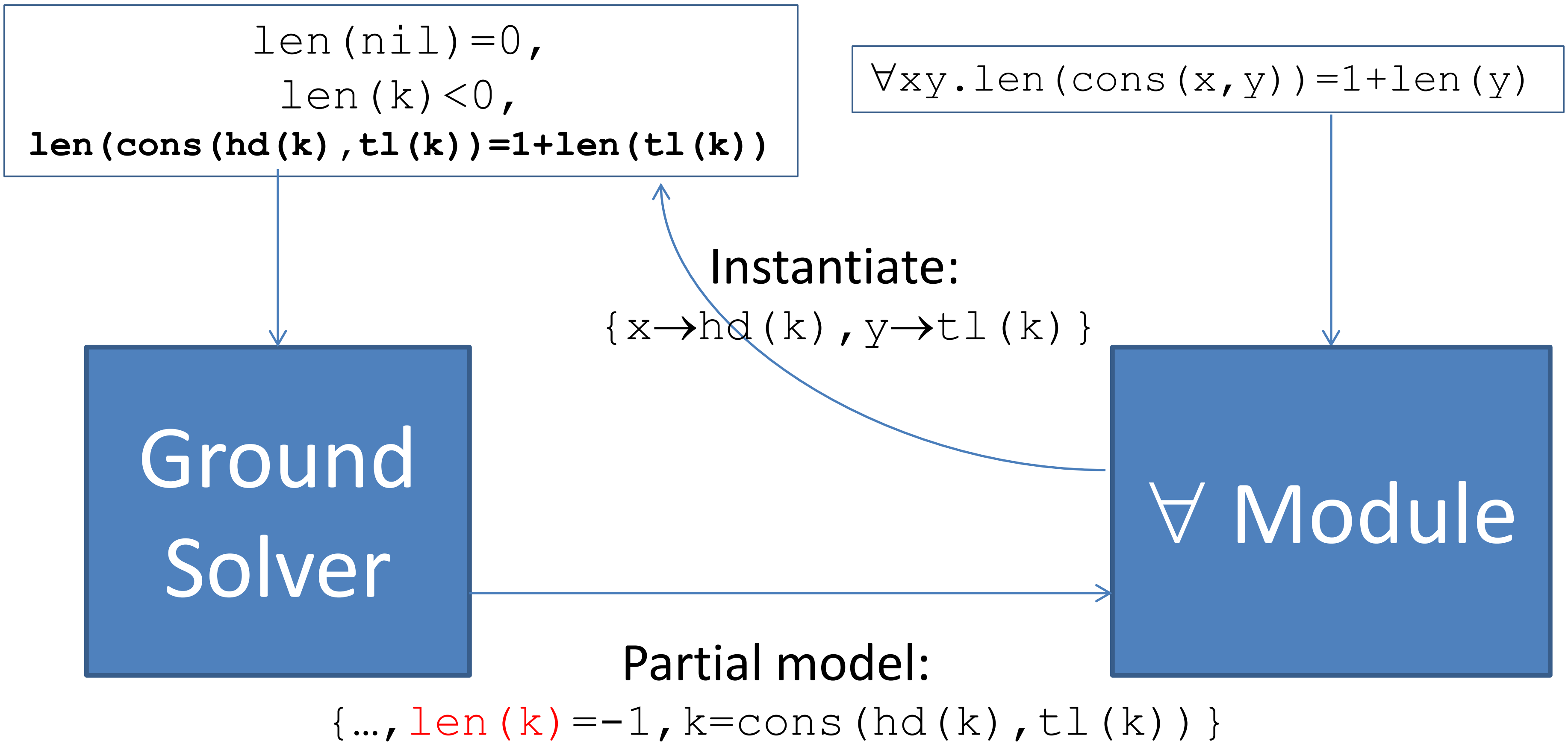
Example #2



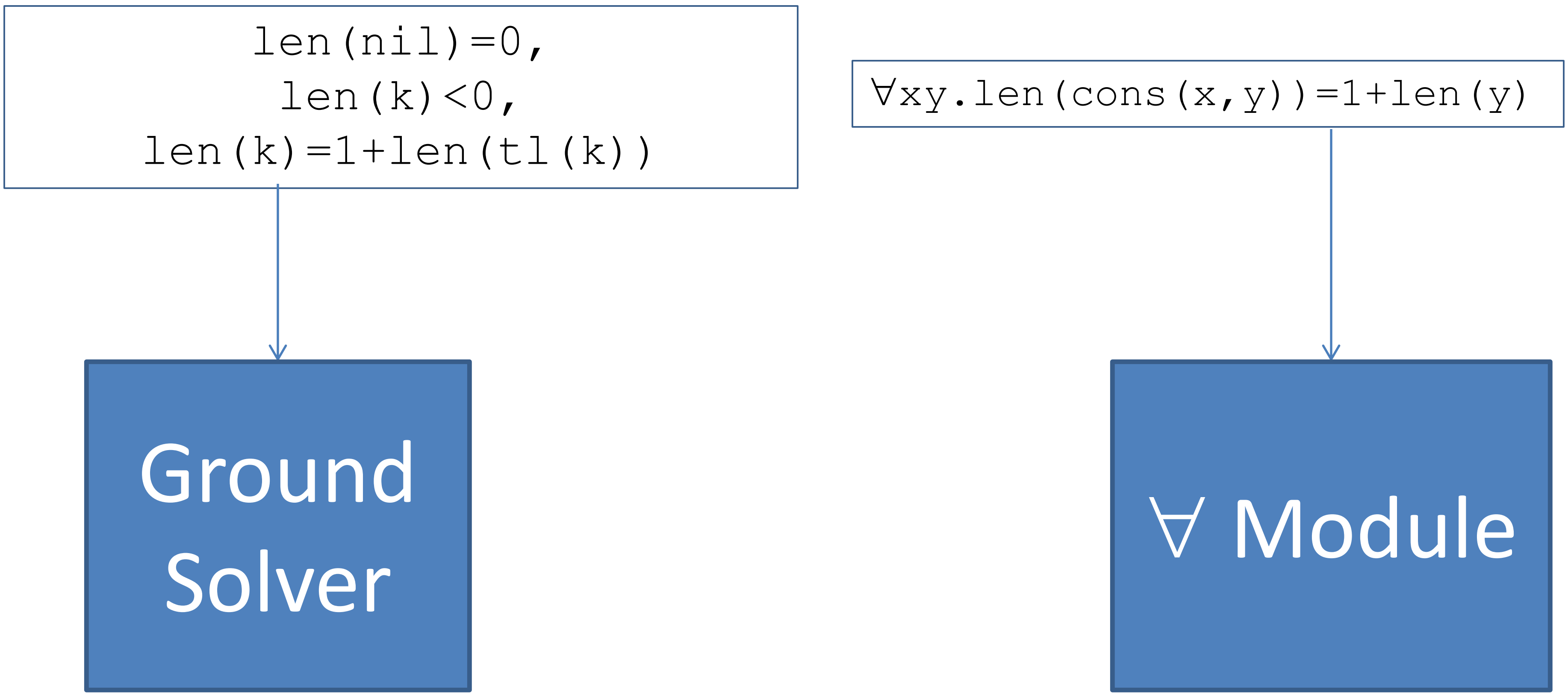
Example #2


len(nil)=0,
len(k)<0
∀xy.len(cons(x,y))=1+len(y)
Ground Solver
∀ Module
Partial model:
{…,len(k)=-1,k=cons(hd(k),tl(k))}

Example #2

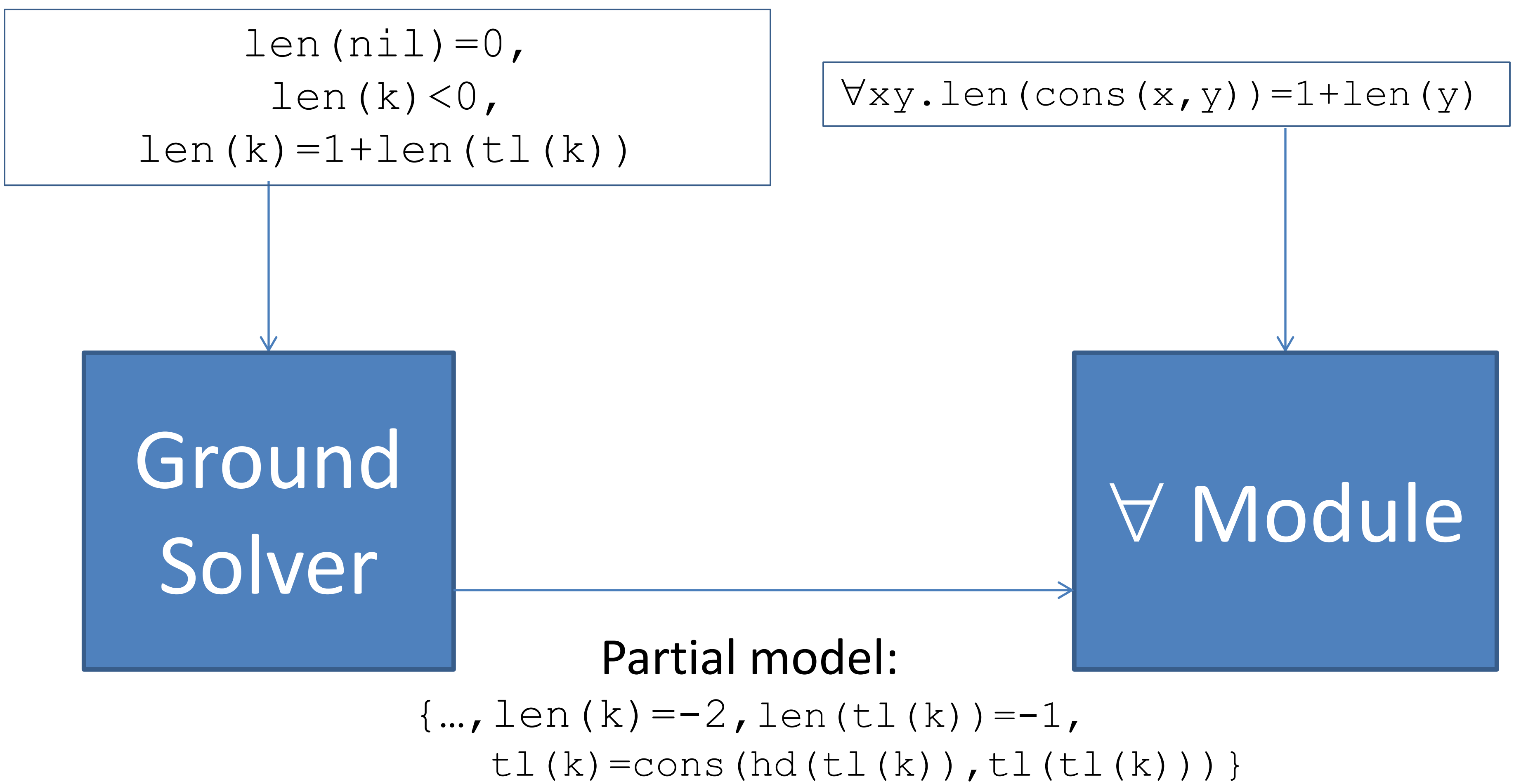

len(nil)=0,
len(k)<0,
len(cons(hd(k),tl(k))=1+len(tl(k))
∀xy.len(cons(x,y))=1+len(y)
Instantiate:
{x→hd(k),y→tl(k)}
Ground Solver
∀ Module
Partial model:
{…,len(k)=-1,k=cons(hd(k),tl(k))}

Example #2



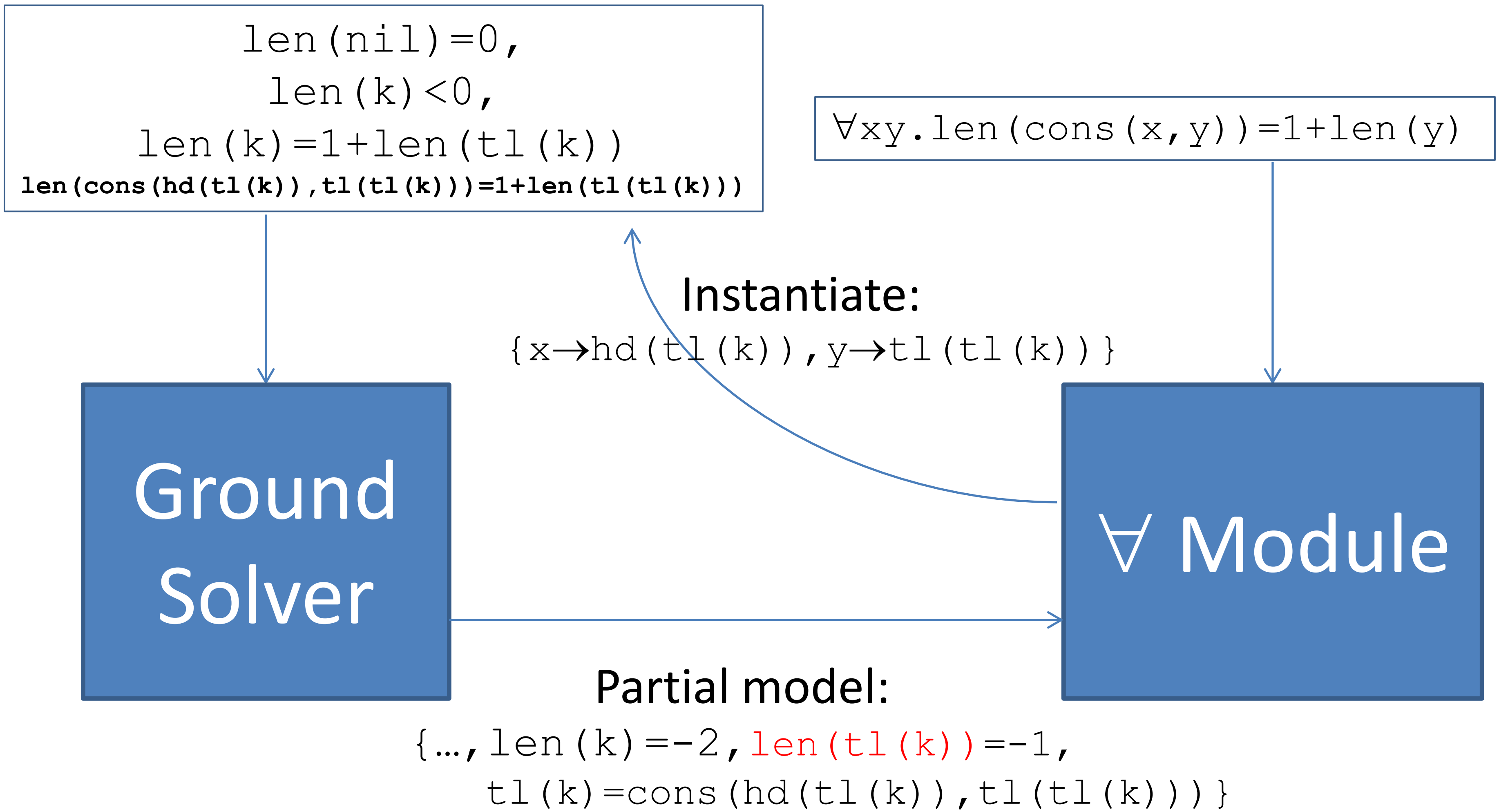
len(nil)=0,
len(k)<0,
len(k)=1+len(tl(k))
∀xy.len(cons(x,y))=1+len(y)
Ground Solver
∀ Module

Example #2



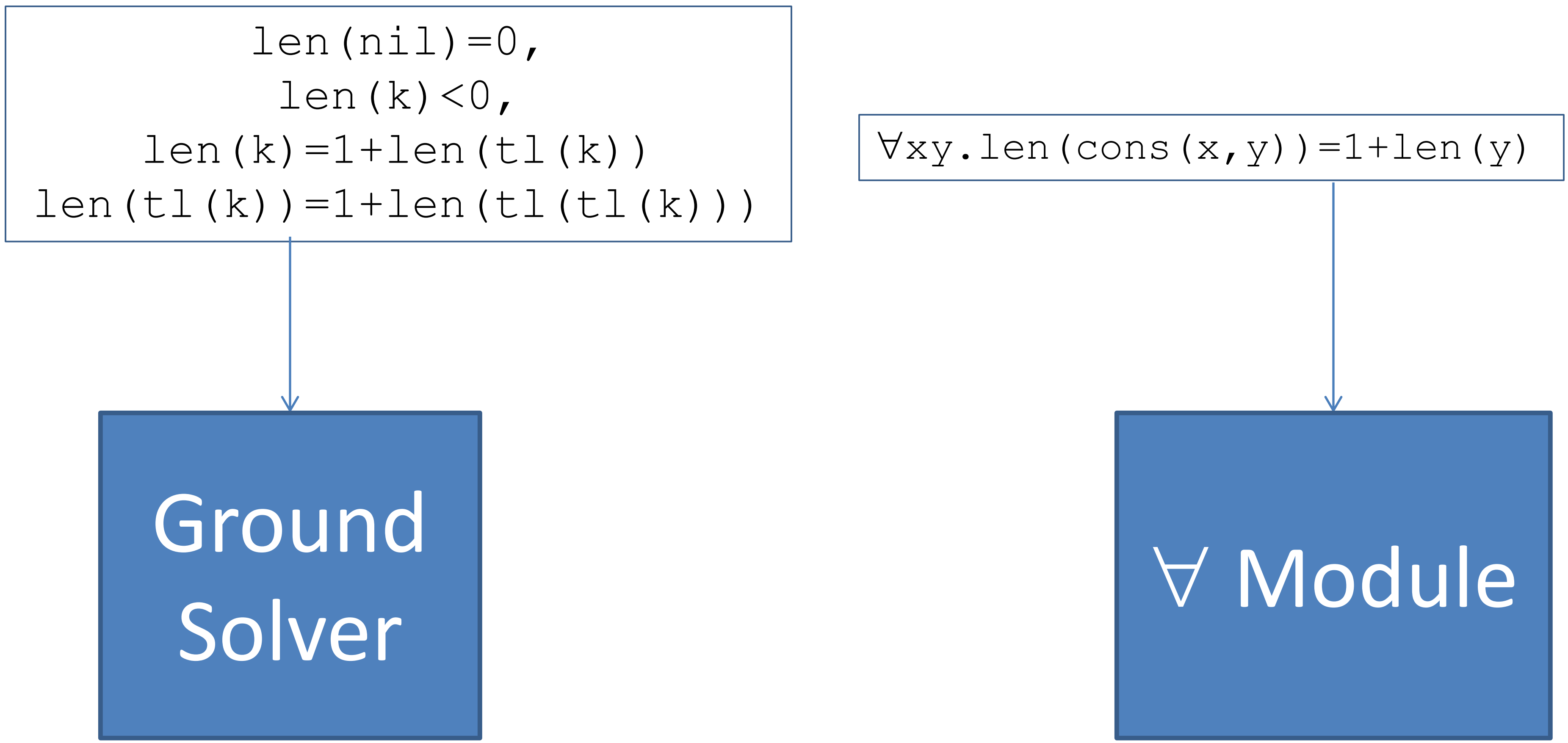
len(nil)=0,
len(k)<0,
len(k)=1+len(tl(k))
∀xy.len(cons(x,y))=1+len(y)
Ground Solver
∀ Module
Partial model:
{…,len(k)=-2,len(tl(k))=-1,
tl(k)=cons(hd(tl(k)),tl(tl(k)))}

Example #2

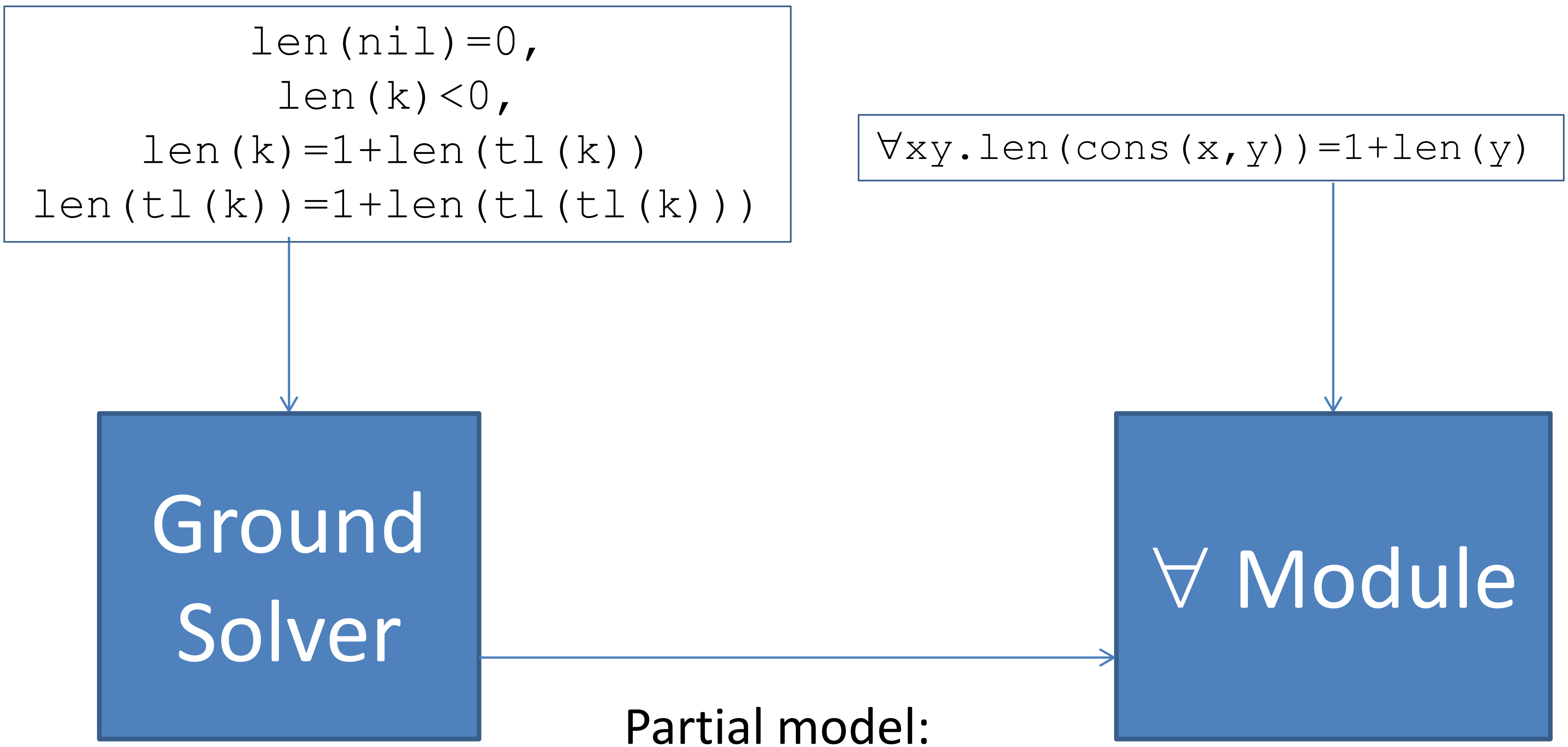


len(nil)=0,
len(k)<0,
len(k)=1+len(tl(k))
len(cons(hd(tl(k)),tl(tl(k)))=1+len(tl(tl(k)))
∀xy.len(cons(x,y))=1+len(y)
Instantiate:
{x→hd(tl(k)),y→tl(tl(k))}
Ground Solver
∀ Module
Partial model:
{…,len(k)=-2,len(tl(k))=-1,
tl(k)=cons(hd(tl(k)),tl(tl(k)))}

Example #2

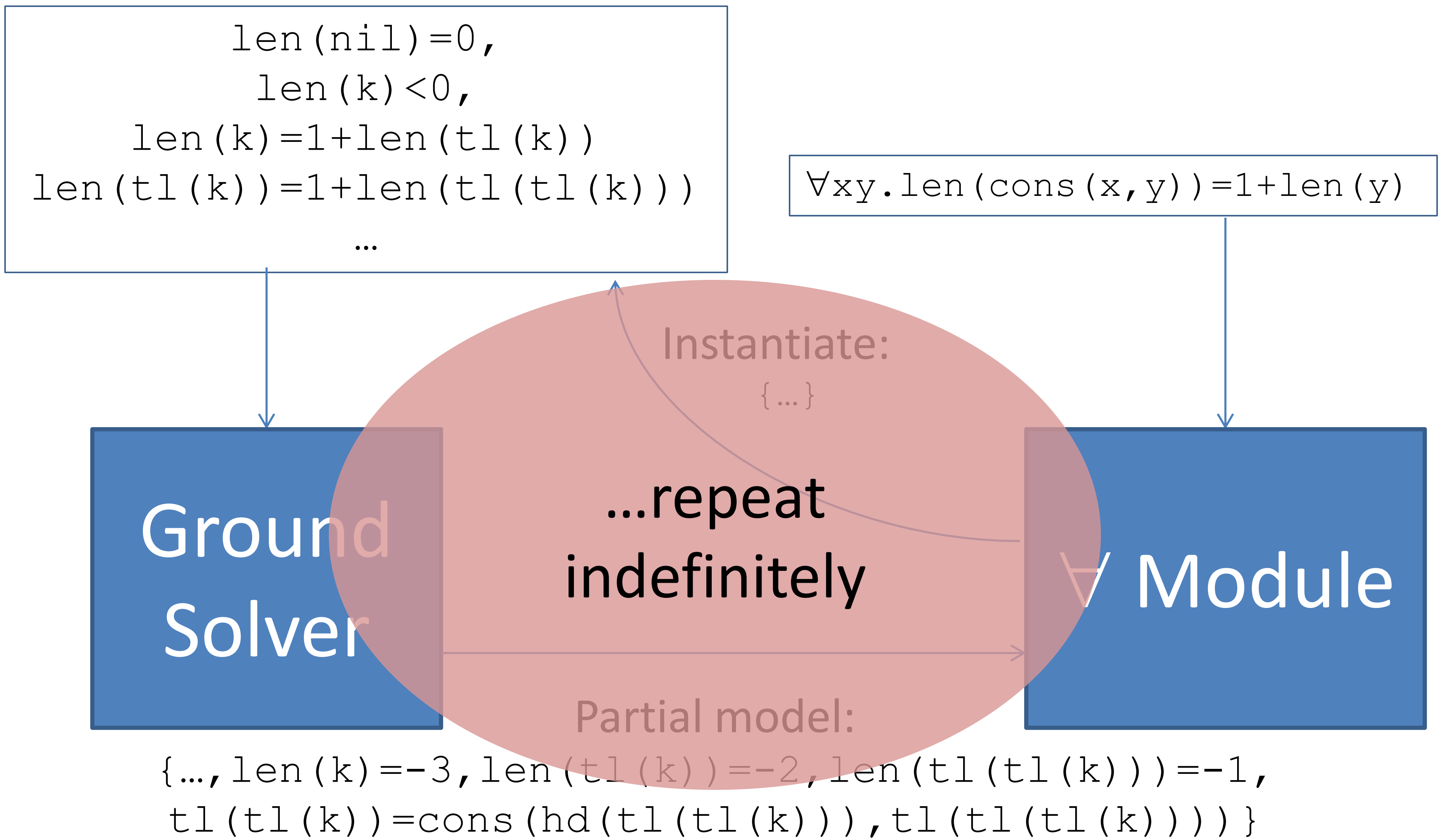

len(nil)=0,
len(k)<0,
len(k)=1+len(tl(k))
len(tl(k))=1+len(tl(tl(k)))
∀xy.len(cons(x,y))=1+len(y)
Ground Solver
∀ Module

Example #2



{…,len(k)=-3,len(tl(k))=-2,len(tl(tl(k)))=-1,
tl(tl(k))=cons(hd(tl(tl(k))),tl(tl(tl(k))))}

Example #2



len(nil)=0,
len(k)<0,
len(k)=1+len(tl(k))
len(tl(k))=1+len(tl(tl(k)))
…
∀xy.len(cons(x,y))=1+len(y)
Instantiate:
{…}
Ground Solver
…repeat indefinitely
∀ Module
Partial model:
{…,len(k)=-3,len(tl(k))=-2,len(tl(tl(k)))=-1,
tl(tl(k))=cons(hd(tl(tl(k))),tl(tl(tl(k))))}

Challenge: Inductive Reasoning

- This example requires induction
- Existing techniques
  - Within inductive theorem provers:
    - ACL2 [Chamarthi et al 2012]
    - Hipspec [Claessen et al 2013]
    - Isaplanner [Johansson et al 2010]
    - Zeno [Sonnex et al 2012]
  - Induction as preprocessing step to SMT solver:
    - Dafny [Leino 2012]
- No SMT solvers support induction *natively*

$\Rightarrow$ Until now, in CVC4

Solution: Inductive Strengthening

- Given negated conjecture:

$$\neg \forall \texttt{x.len(x)} \geq 0$$

- Assume property does not for fresh k:

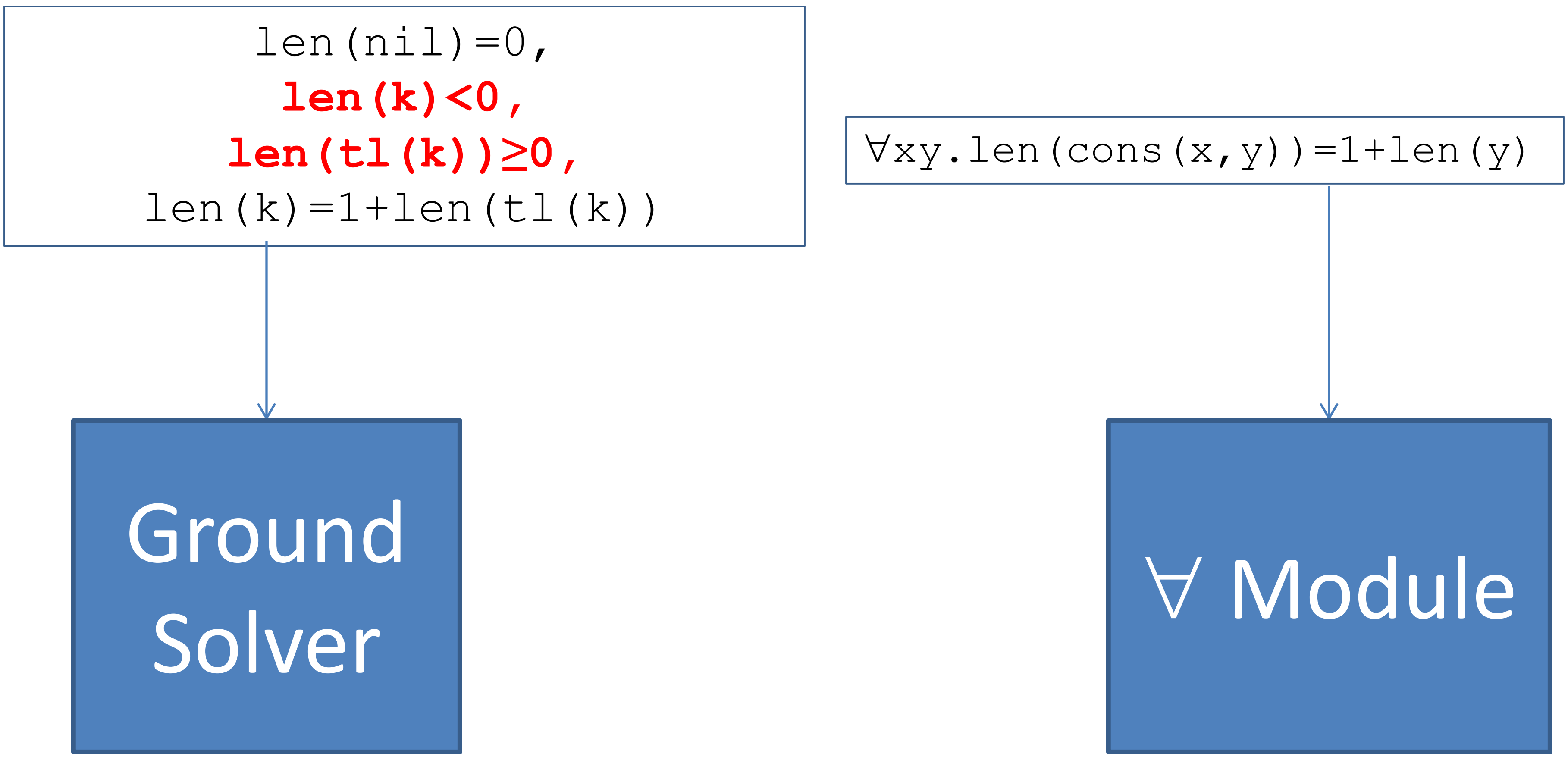
$$\neg\ \texttt{len(k)} \geq 0$$

AND

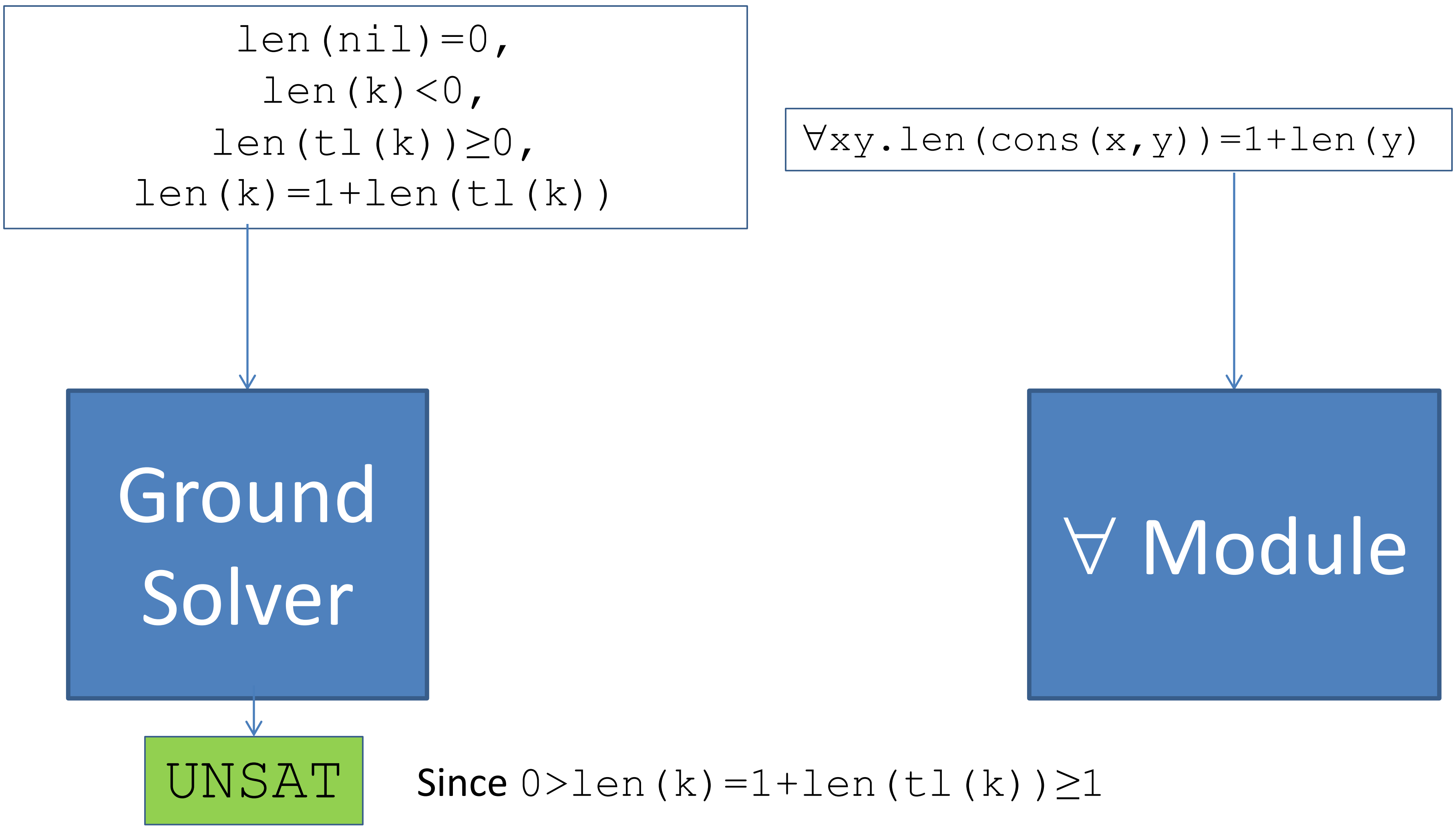
- Assume k is the *smallest* CE to property:

$$\texttt{k=cons(hd(k),tl(k))} \Rightarrow \texttt{len(tl(k))} \geq 0$$

Example #2: revised



Example #2: revised



Skolemization with Inductive Strengthening

- General form:

$$\boxed{\forall x. P(x) \lor (\neg P(k) \land \forall y. R(y,k) \Rightarrow P(y))}$$

  - For well-founded relation R
- Extends for multiple variables
- Common examples in SMT:
  - (Weak) structural induction on inductive datatypes
    - Assume property holds for direct children of k of same type
  - (Weak) well-founded induction on integers
    - Assume property holds for (k-1), with base case 0

Challenge: Subgoal Generation

- Unfortunately, inductive strengthening is not enough
- Consider conjecture:

```
∀x.len(rev(x))=len(x)
```

  – where `rev` is axiomatized by:

```
rev(nil)=nil,
∀xy.rev(cons(x,y))=app(rev(y),cons(x,nil))
```

- To prove, requires induction, and "subgoals":

```
∀xy.len(app(x,y))=plus(len(x),len(y))
```

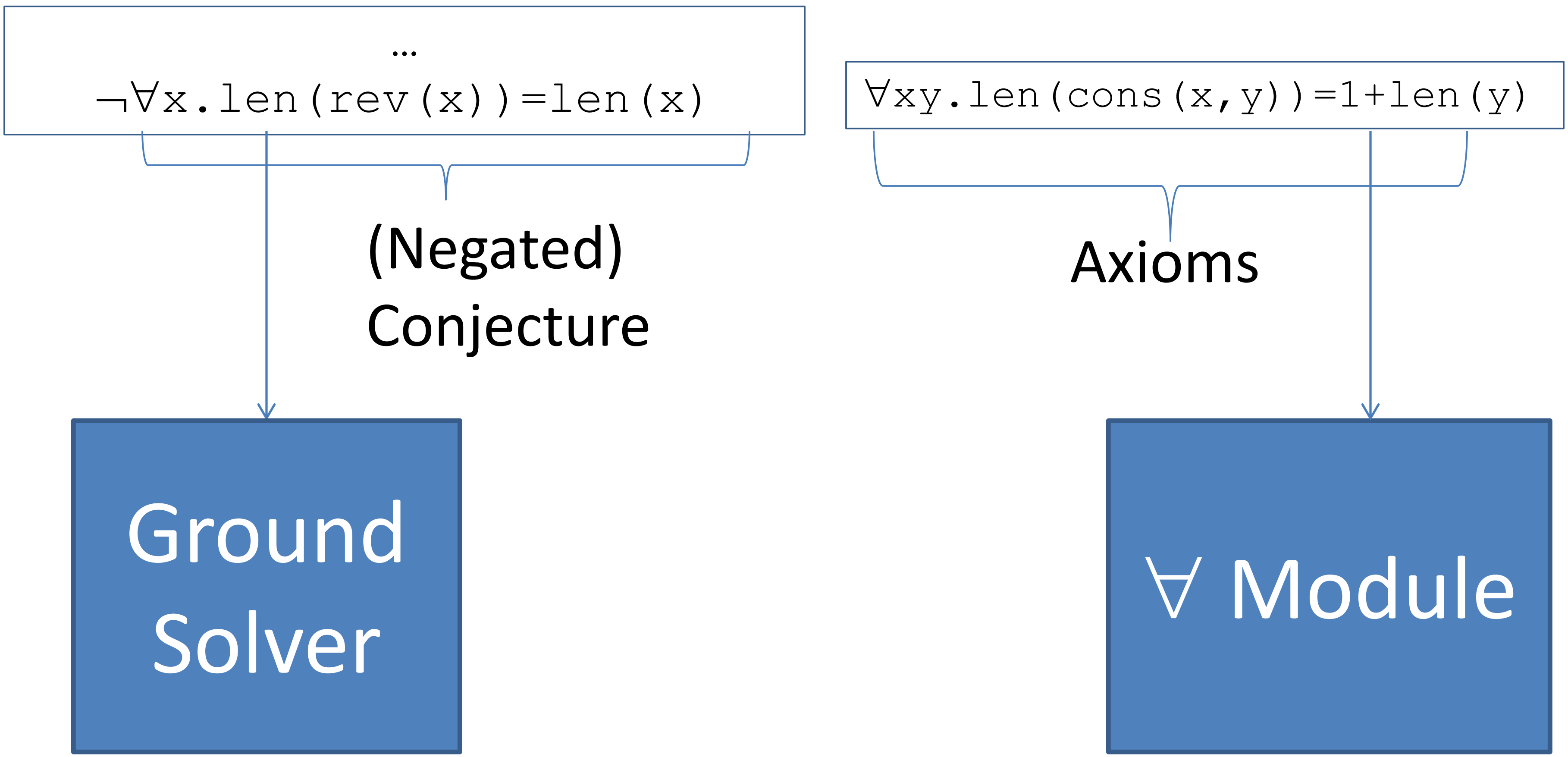
```
∀xy.plus(x,y)=plus(y,x)
```

Generating candidate subgoals

- How to generate necessary subgoals?
  - Idea: Enumerate/prove them in a principled way
    - QuickSpec [Claessen et al 2010]

```
∀x.len(x)=Z
∀x.len(x)=S(Z)
∀x.app(x,nil)=nil
∀x.app(x,nil)=x
∀x.app(x,nil)=cons(0,x)
…
∀xy.plus(x,y)=plus(x,0)
∀xy.plus(x,y)=plus(y,x)
…
∀xy.len(app(x,y))=plus(len(x),len(y))
…
```

Subgoal Generation in SMT



Subgoal Generation in SMT

…

$\neg\forall x.$`len(rev(x))=len(x)`

$\forall xy.$`len(cons(x,y))=1+len(y)`

$\neg\forall x.$`len(x)=Z`	$\vee$	$\forall x.$`len(x)=Z`
$\neg\forall x.$`app(x,nil)=x`	$\vee$	$\forall x.$`app(x,nil)=x`
…	$\vee$	…
$\neg\forall xy.$`len(app(x,y))=plus(len(x),len(y))`	$\vee$	$\forall xy.$`len(app(x,y))=plus(len(x),len(y))`

Ground Solver

- Enumerate subgoals

  For each, either:

  (a) it has a c.e.,

  (b) holds universally

$\forall$ Module

$\Rightarrow$ Use "splitting-on-demand"

Subgoal Generation in SMT

...

¬∀x.len(rev(x))=len(x)

∀xy.len(cons(x,y))=1+len(y)

¬∀x.len(x)=Z ∨ ∀x.len(x)=Z

¬∀x.app(x,nil)=x ∨ ∀x.app(x,nil)=x

... ∨ ...

¬∀xy.len(app(x,y))=plus(len(x),len(y)) ∨ ∀xy.len(app(x,y))=plus(len(x),len(y))

Ground Solver

∀ Module

Subgoal Generation in SMT

…

$\mathtt{len(rev(k_0))} \neq \mathtt{len(k_0)},$

$\mathtt{len(rev(tl(k_0)))=len(tl(k_0))}$

$\forall xy.\mathtt{len(cons(x,y))=1+len(y)}$

$\mathtt{len(k_1)} \neq \mathtt{Z}, \mathtt{len(tl(k_1))=Z} \quad \vee \quad \forall x.\mathtt{len(x)=Z}$

$\mathtt{app(k_2,nil)} \neq \mathtt{k_2}, \mathtt{app(tl(k_2),nil)=tl(k_2)} \quad \vee \quad \forall x.\mathtt{app(x,nil)=x}$

$\ldots \quad \vee \quad \ldots$

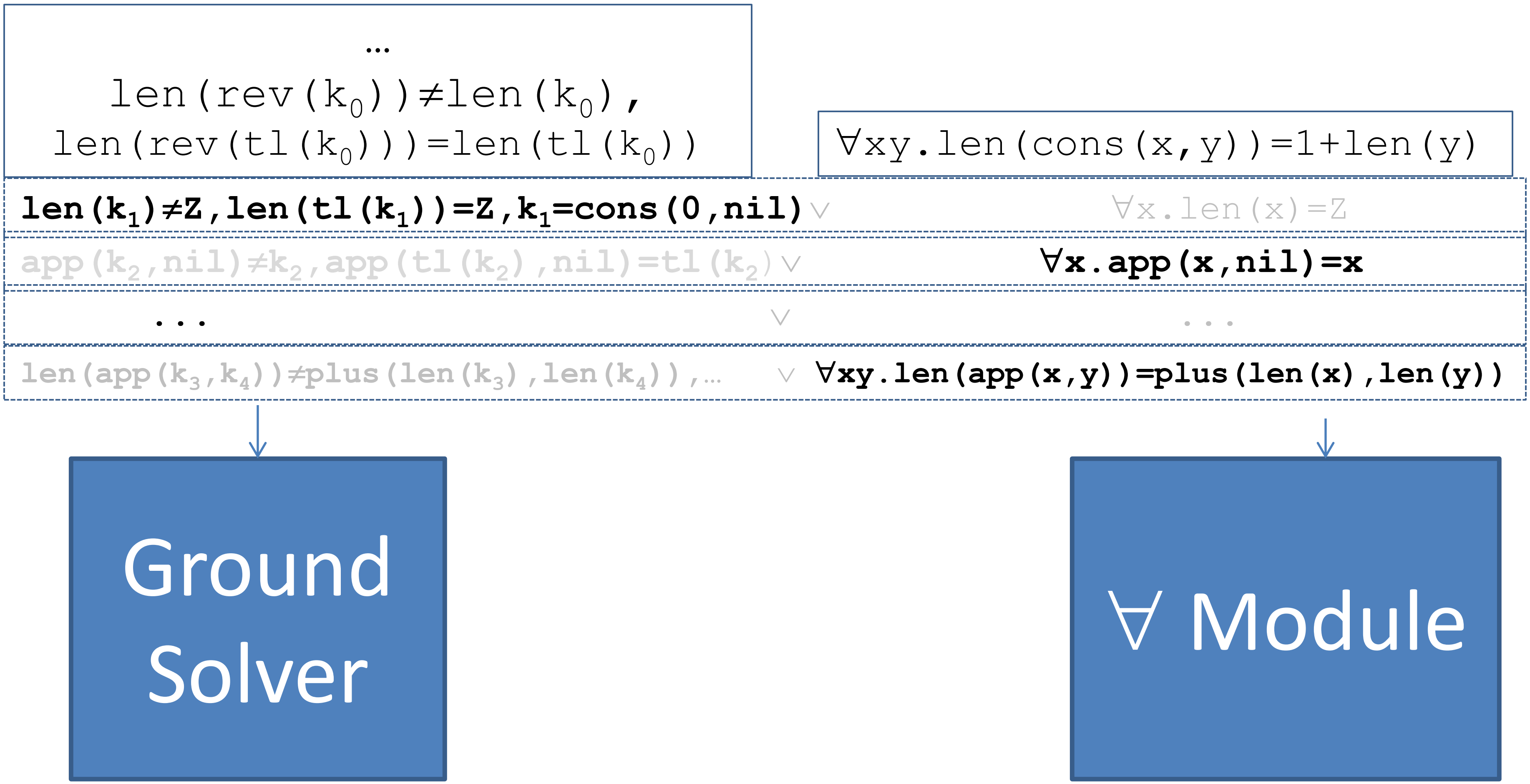
$\mathtt{len(app(k_3,k_4))} \neq \mathtt{plus(len(k_3),len(k_4))}, \ldots \quad \vee \quad \forall xy.\mathtt{len(app(x,y))=plus(len(x),len(y))}$

Ground Solver

∀ Module

- Each subgoal can be inductively strengthened
  - I.e. holds for a smallest counterexample

Subgoal Generation in SMT



- When negated s.g. is SAT, deduce ground information
- When negated s.g. is UNSAT, it can be used as axiom

Subgoal Generation : Challenges

- Generate subgoals by enumeration
- Main challenge: scalability
- Keys to success:
  - Generate s.g. in a fair manner (smaller s.g. first)
  - Filter out s.g. that are not useful

Subgoal Filtering

- Given: $\boxed{\forall x.\texttt{len(rev(x))=len(x)}}$
- Filtering based on "active" conjectures:

  🚫 $\forall$`xy.count(x,y)=count(rev(x),y)`
  - Irrelevant, since conjecture is not related to "`count`"
- Filtering based on canonicity:

  🚫 $\forall$`x.len(x)=len(app(x,nil))`
  - Redundant, since we know $\forall$`x.x=app(x,nil)`
- Filtering based on counterexamples:

  🚫 $\forall$`x.len(x)=len(app(x,x))`
  - False, since we know `len(x)`$\neq$`len(app(x,x))` for `x`$\neq$`nil`

$\Rightarrow$ Using techniques, typically can remove >95% subgoals

Experiments : Benchmarks

- Four benchmark sets:
  1. Isaplanner [Johansson et al 2010]
     - List, Nats, Trees, (some) higher-order functions
  2. Clam [Ireland 1996]
     - Lists, Nats, Sets
       - Designed specifically to require subgoals
  3. Hipspec [Claessen et al 2013]
     - Lists, Nats
       - e.g. : sum of n cubes is square of nth triangle number
  4. Leon
     - Amortized Queues, Binary search trees, Leftist Heaps

Experiments : Encodings

1. Base encoding (**dt**), e.g. defined plus as:

```
∀x.plus(Z,x)=x
∀xy.plus(S(x),y)=S(plus(x,y))
```

2. Theory encoding (**dtt**)
   – Rephrase axioms/conjectures in terms of "+"
3. Theory-isomorphism encoding (**dti**)
   – Keep encoding, provide mappings to theory symbols:
     • Injection "`toInt`" from dt to int, with axiom:

$$\left.\boxed{\texttt{∀xy.toInt(plus(x,y))=toInt(x)+toInt(y)}}\right\} A$$

$\Rightarrow$ *2,3 allow SMT solver to leverage theory reasoning*

– Thus, we get subgoals for "free", e.g.:

$$A \models_T \boxed{\texttt{∀xy.plus(x,y)=plus(y,x)}}$$

Results : SMT solvers

dt	**Isaplan**	**Clam**	**HSpec**	**Leon**
z3	16	0	0	6
cvc4	15	0	0	7
cvc4+i	68	16	12	29
cvc4+ig	75	38	17	34
dtt	**Isaplan**	**Clam**	**HSpec**	**Leon**
z3	35	9	2	9
cvc4	34	7	2	8
cvc4+i	64	16	11	37
cvc4+ig	67	25	12	39
dti	**Isaplan**	**Clam**	**HSpec**	**Leon**
z3	35	8	2	9
cvc4	34	8	2	9
cvc4+i	76	25	15	41
cvc4+ig	80	39	18	42
Total	*85*	*50*	*26*	*45*

- 300 second timeout

cvc4+i:
with induction

cvc4+ig:
with induction
+subgoal gen.

Results: Subgoal Generation

- With subgoals, solved +37 for **dti** encoding
  - Only solved +1 when filtering turned off
- Most subgoals were small: term size $\leq$ 3
- Can find simpler goals than by manual inspection:
  - For conjecture:

$$\forall xy.\mathtt{count(x,y)=count(insort(x),y)}$$

  - We thought it would require:

$$\forall xy.\mathtt{count(ins(y,x),y)=S(count(x,y))},$$
$$\forall xyz.\mathtt{y}\neq\mathtt{z}\Rightarrow\mathtt{count(ins(y,x),z)=count(x,z)}$$

  - CVC4 found:

$$\forall xyz.\mathtt{count(ins(y,x),z)=count(cons(y,x),z)}$$

$\Rightarrow$ *Suffices to prove conjecture, which CVC4 did fully automatically*

Comparison with Other Provers

	Isaplan	**Clam**	**HSpec**	**Leon**
cvc4+ig (dti)	80	39	18	42
ACL2	73			
Clam		41		
Dafny	45			
Hipspec	80	47	(26)	
Isaplanner	43			
Zeno	82	21		
Total	*85*	*50*	*26*	*45*

- Translated/evaluated in previous studies
- Tools tend to perform well on benchmarks they are tuned for
  - CVC4 competitive with state-of-the-art inductive theorem provers

Summary

- Techniques for Induction in CVC4
- Best performance by making use of:
  - Theory reasoning (dti encoding)
  - Subgoal generation
- Competitive with inductive theorem provers

Future Work

- Improvements to subgoal generation
  - Filtering heuristics
  - User-guided/interactive approaches
- Incorporate more induction schemes
- Completeness criteria
  - Identify cases approach is guaranteed to succeed
- Standard format for inductive theorem provers
- Applications:
  - Use within Leon verification tool (EPFL)
  - Synthesis of recursive functions

Thanks!

- CVC4 publicly available:
  - http://cvc4.cs.nyu.edu/downloads/
  - Induction techniques:
    - Enabled by "`--quant-ind`"
    - More details, see VMCAI 2015 paper

- Questions?