Automated Discovery of Invertibility Conditions via Syntax-Guided Synthesis

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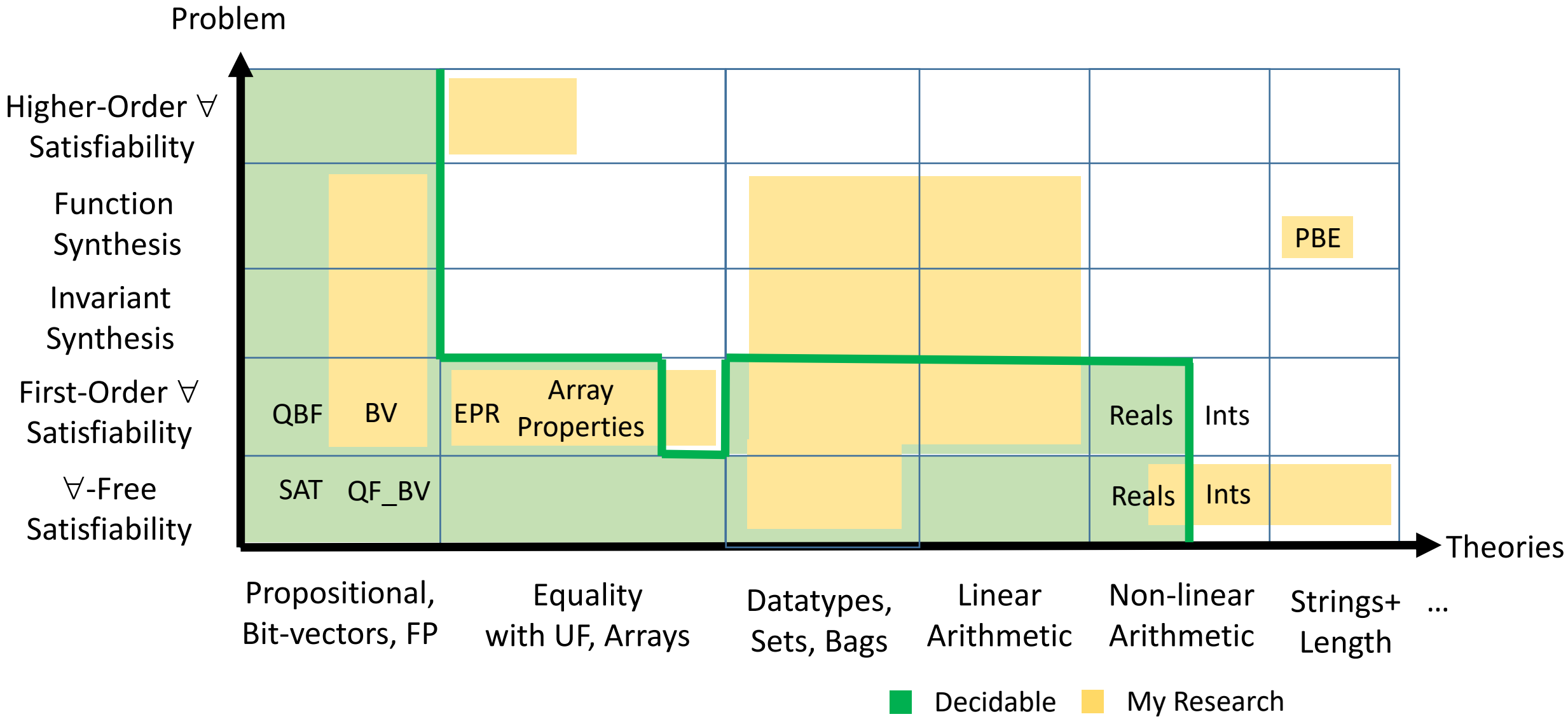
My Research

- Satisfiability Modulo Theories (SMT) Solvers
  - Fully automated reasoners with many applications
    - Verification, Synthesis, Symbolic Execution, Theorem Proving, Security Analysis

- Development of the SMT solver cvc5
  - Open source, available at : https://cvc5.github.io/

- Acknowledgements:
  - Cesare Tinelli, Clark Barrett, Haniel Barbosa, Andres Noetzli, Aina Niemetz, Mathias Preiner
  - Rest of cvc5 development team (past and present)
  - Viktor Kuncak

SMT Solvers: Standalone Tools for Many Problems



In this Talk:

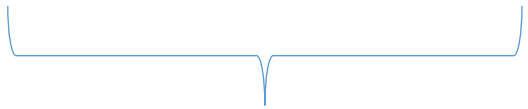
- Using *Syntax-Guided Synthesis* (SyGuS) to aid SMT solver development

- How we use SyGuS for developing cvc5:
  - Discovering rewrite rules **[Noetzli et al SAT 2019]**
  - Automatic test case generation
  - Discovering Invertibility Conditions for Bit-Vectors **[Niemetz et al CAV 2018]**
  - Discovering Invertibility Conditions for *Floating Points* **[Brain et al CAV 2019]**

How does Syntax-Guided Synthesis work?

Synthesis Conjectures

$$\exists f.\forall x.P(f,x)$$



There exists a function `f` for which property `P` holds for all `x`

Syntax-Guided Synthesis Conjectures Modulo T

$$\exists f.\forall x.P(f,x)$$

spec(f)

There exists a function `f` for which property `P` holds for all `x`

$$A \to A+A \mid -A \mid x \mid y \mid 0 \mid 1 \mid ite(B,A,A)$$
$$B \to B \wedge B \mid \neg B \mid A=A \mid A \geq A \mid \bot$$

syntax(f)

The body of `f` is generated by the above grammar with start symbol `A`

$\Rightarrow$ *Syntax-guided synthesis "SyGuS"* **[Alur et al 2013]**

Enumerative Cex-Guided Inductive Synthesis (CEGIS)

```
syntax(f):
A->A+A|-A|x|y|0|1|ite(B,A,A)
B->B∧B|¬B|A=A|A≥A|⊥
```

```
spec(f):
∀xy.f(x,y)=f(y,x)+f(0,1)
```

Solution *Enumerator*

Solution Verifier

[Solar-Lezama 2006, Alur et al 2013]

Enumerative Cex-Guided Inductive Synthesis (CEGIS)

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syntax(f):
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∀xy.f(x,y)=f(y,x)+f(0,1)
```

Solution
Enumerator

Solution
Verifier

`f:=λxy.x?`

Enumerative Cex-Guided Inductive Synthesis (CEGIS)

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syntax(f):
A->A+A|-A|x|y|0|1|ite(B,A,A)
B->B∧B|¬B|A=A|A≥A|⊥
```

```
spec(f):
∀xy.f(x,y)=f(y,x)+f(0,1)
```

Solution *Enumerator*

Solution Verifier

`f(0,1)=f(1,0)+f(0,1)`

`f:=λxy.x?`

..counterexample: spec does not hold for **`(x,y)=(0,1)`**

Enumerative Cex-Guided Inductive Synthesis (CEGIS)

```
syntax(f):
A->A+A|-A|x|y|0|1|ite(B,A,A)
B->B∧B|¬B|A=A|A≥A|⊥
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spec(f):
∀xy.f(x,y)=f(y,x)+f(0,1)
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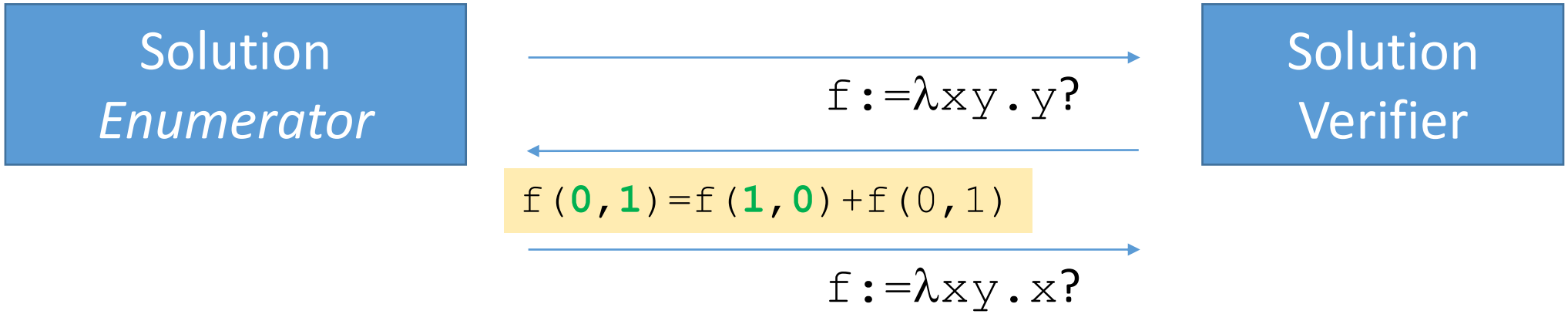

…new candidate λxy.y satisfies **(x,y)=(0,1)**

Enumerative Cex-Guided Inductive Synthesis (CEGIS)

```
syntax(f):
A->A+A|-A|x|y|0|1|ite(B,A,A)
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spec(f):
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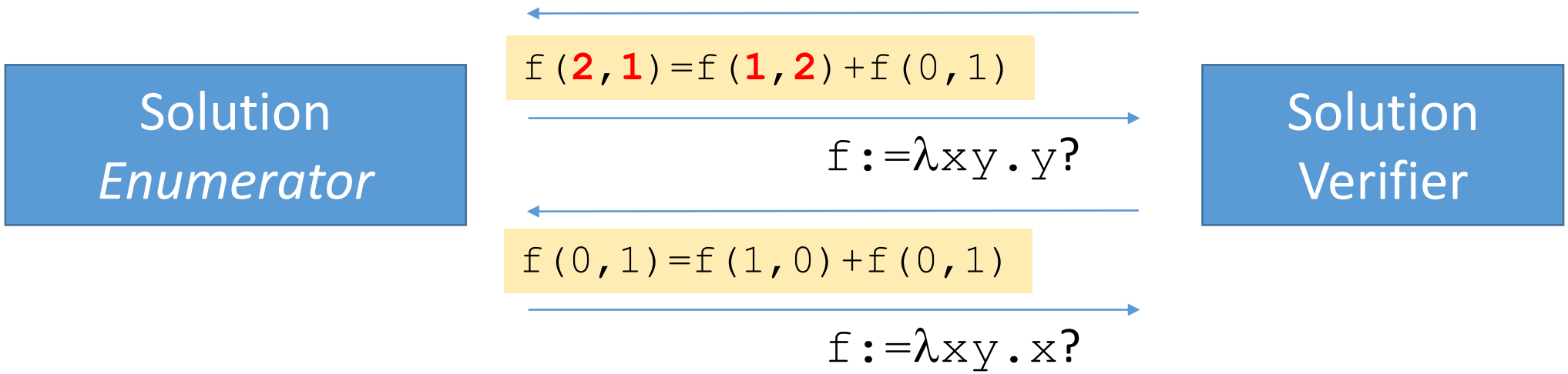
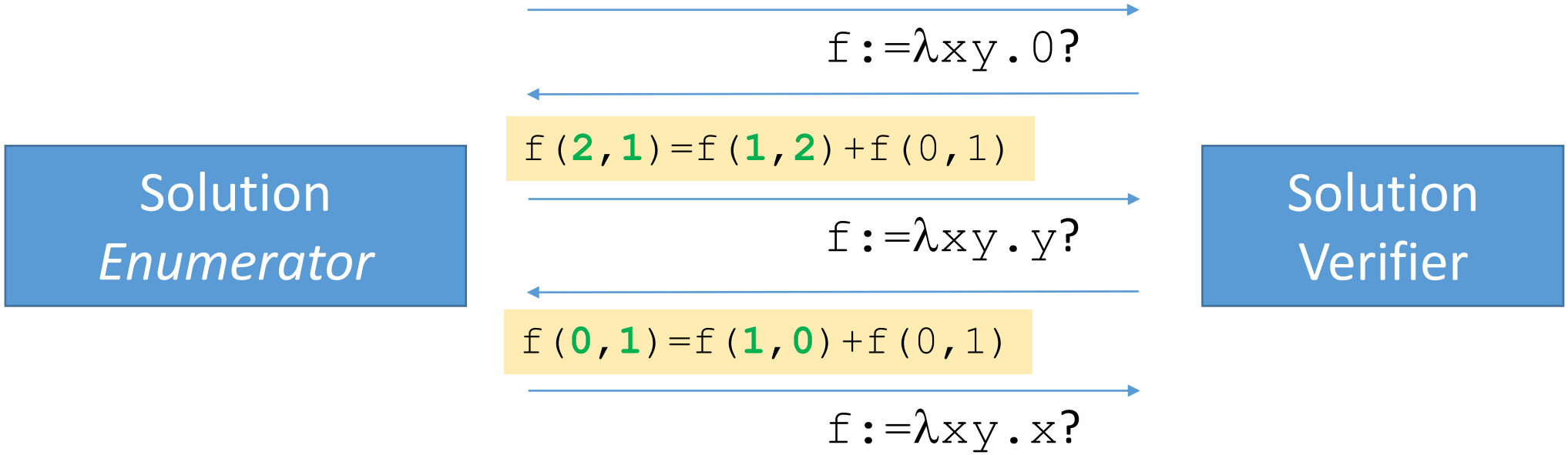


…counterexample: spec does not hold for **(x,y)=(2,1)**

Enumerative Cex-Guided Inductive Synthesis (CEGIS)

```
syntax(f):
A->A+A|-A|x|y|0|1|ite(B,A,A)
B->B∧B|¬B|A=A|A≥A|⊥
```

```
spec(f):
∀xy.f(x,y)=f(y,x)+f(0,1)
```



…new candidate λxy.0 satisfies **(x,y)=(0,1),(2,1)**

Enumerative Cex-Guided Inductive Synthesis (CEGIS)

syntax(f):
A->A+A|-A|x|y|0|1|ite(B,A,A)
B->B∧B|¬B|A=A|A≥A|⊥

spec(f):
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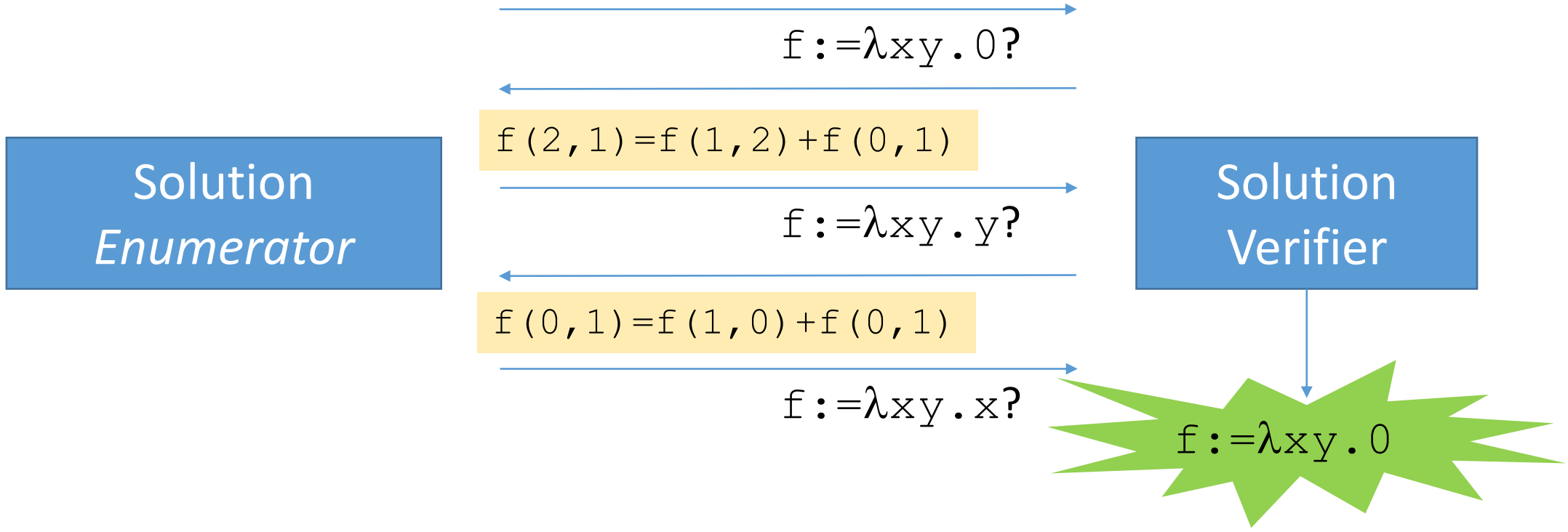
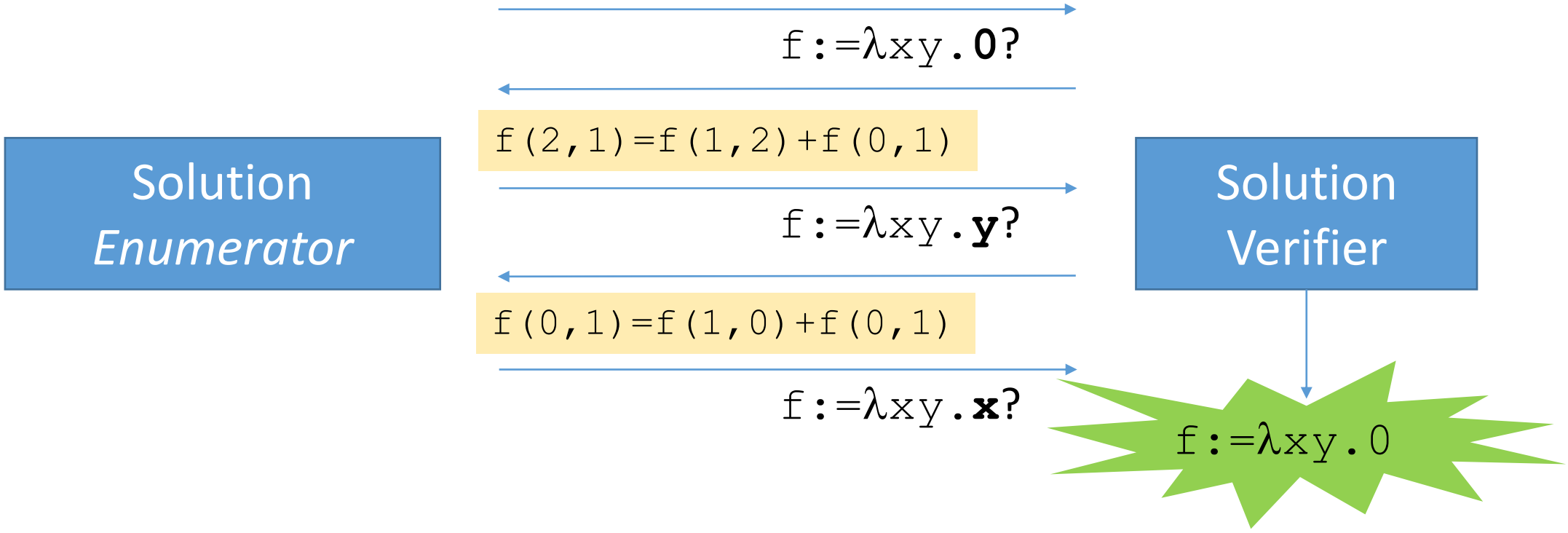


…new candidate λxy.0 has no counterexamples wrt **spec(f)**

Enumerative Cex-Guided Inductive Synthesis (CEGIS)

syntax(f):
A->A+A|-A|x|y|0|1|ite(B,A,A)
B->B∧B|¬B|A=A|A≥A|⊥

spec(f):
∀xy.f(x,y)=f(y,x)+f(0,1)



⇒ Terms **x**,**y**,**0**,… are a (fair) enumeration of terms generated by **syntax(f)**

CEGIS using SMT solvers

```
syntax(f):
A->A+A|-A|x|y|0|1|ite(B,A,A)
B->B∧B|¬B|A=A|A≥A|⊥
```

```
spec(f):
∀xy.f(x,y)=f(y,x)+f(0,1)
```

Solution Enumerator

SMT Solver

Solution Verifier

CEGIS inside an SMT solver

```
syntax(f):
A->A+A|-A|x|y|0|1|ite(B,A,A)
B->B∧B|¬B|A=A|A≥A|⊥
```

```
spec(f):
∀xy.f(x,y)=f(y,x)+f(0,1)
```

SMT Solver
(cvc5)

[Reynolds et al CAV 2015]

Solution Enumerator

Solution Verifier

Using an SMT Solver (cvc5) to *Improve Itself*

1. Rewrite Rule Synthesis

$$\exists t_1 t_2 . \forall x . t_1[x] = t_2[x],\ t_1 \not\to t_2$$

[Noetzli et al SAT 2019]

2. Semantics-guided test case generation

$$\exists Q . \exists_{\leq n} x \in S .\ Q(x)$$

3. Solving Quantified Bit-Vectors (and Floating Points) Using Invertibility Conditions

$$\exists IC . \forall st . IC(s,t) \Leftrightarrow \exists x . x \oplus s \sim t$$

[Niemetz et al CAV 2018,
Brain et al CAV 2019]

…can tackle these problems using syntax-guided synthesis

Application #1:
Rewrite Rule Synthesis

Rewriting is Important for SMT Solving

- SMT solvers use a "rewriter" to put constraints in some normal form
  - E.g. `x+0`$\rightarrow$`x`, `x-y`$\rightarrow$`x+(-1*y)`, `x=x-2`$\rightarrow\bot$

- Having a good rewriter is highly critical to performance
  - In particular, theory of bit-vectors, strings, floating points
    - Single rewrite may make problem go from hard $\rightarrow$ trivial

Rewrite Rules are *Difficult to Implement*

- Hard to find commonly applicable rewrites
  - Analyze problem instances, solver runs
- What rewrites have I not already implemented?
- Time consuming, many lines of code
  - cvc5's BV rewriter ~3500 LOC
  - cvc5's string rewriter >5000 LOC

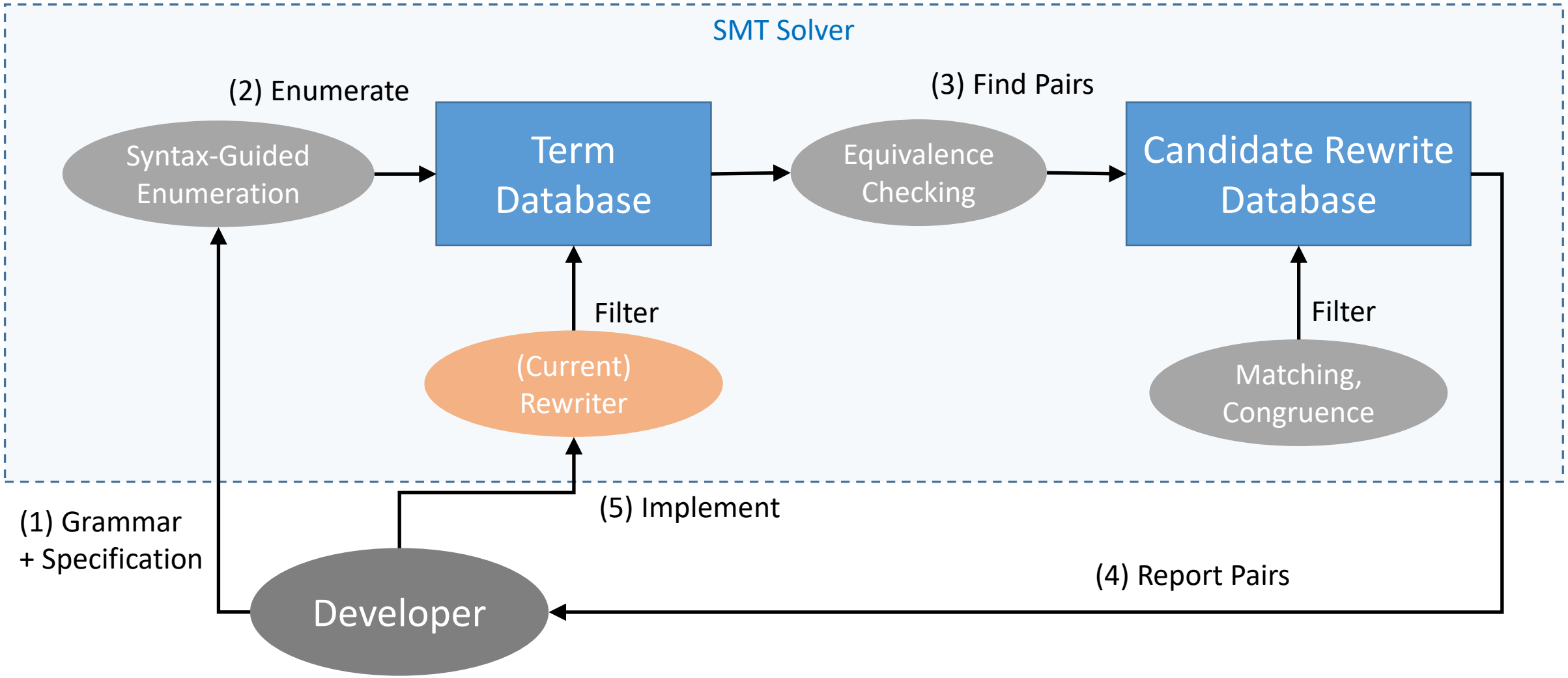
Rewrite Rule Synthesis

- Use the syntax-guided enumeration to assist the developer to implement the solver's rewriter

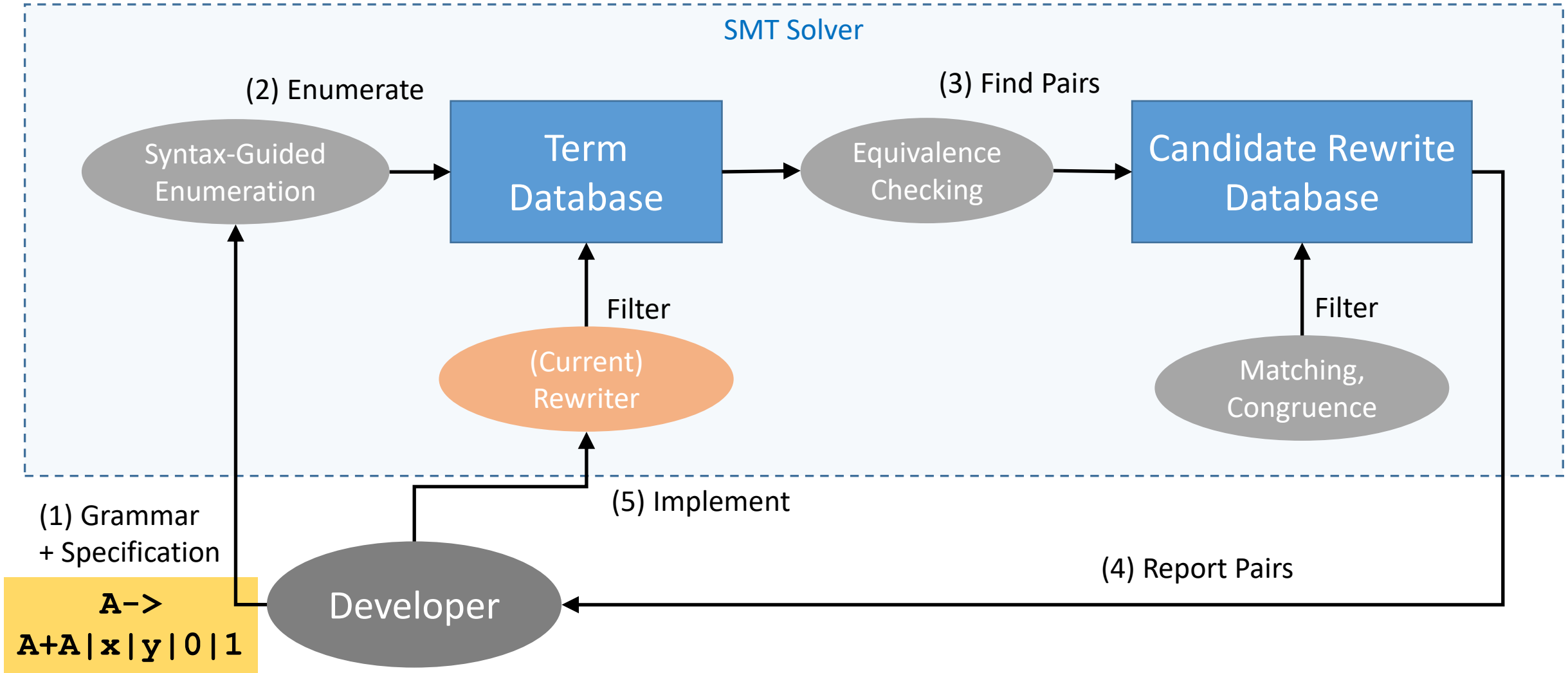
$\Rightarrow$Increase productivity of the developer

$\Rightarrow$Increase confidence in the correctness of the rewriter

(Partially) Automated Rewrite Rule Generation



(Partially) Automated Rewrite Rule Generation



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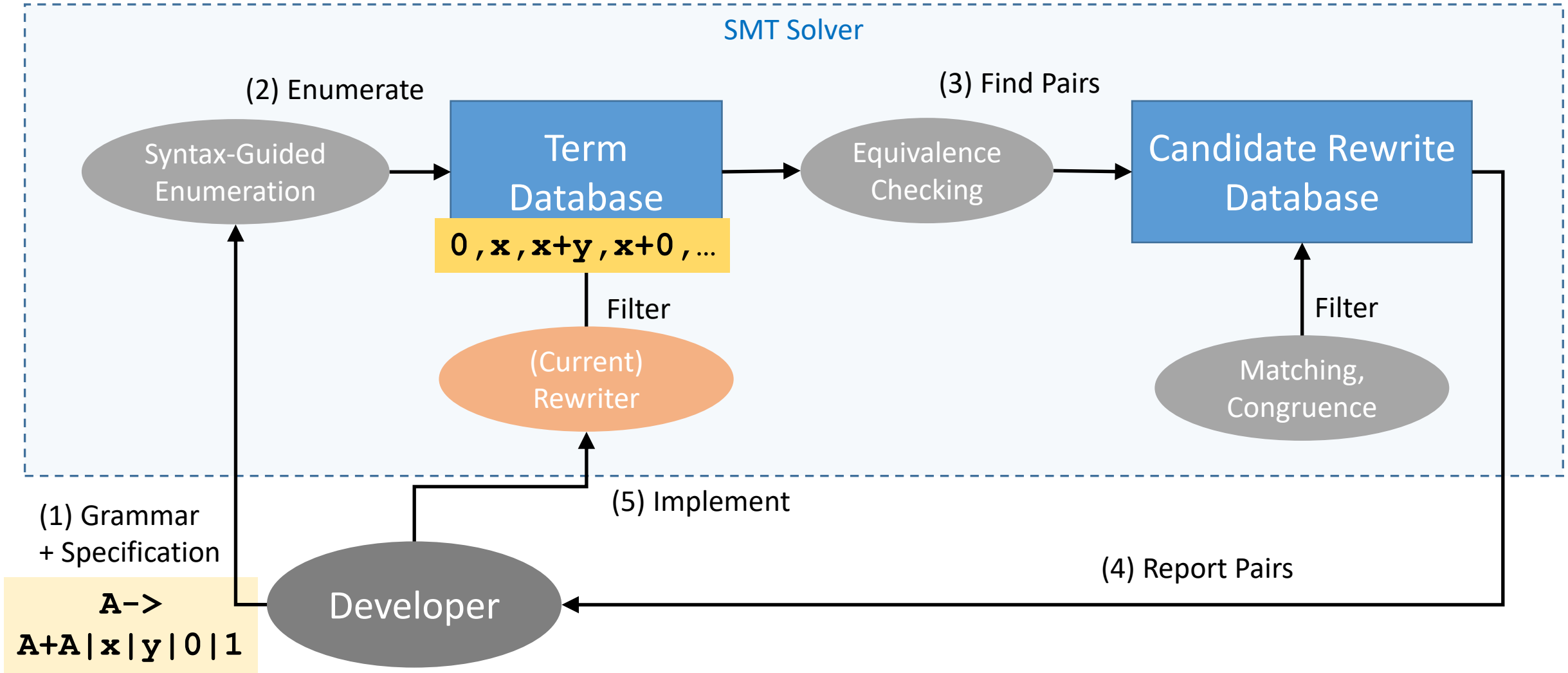
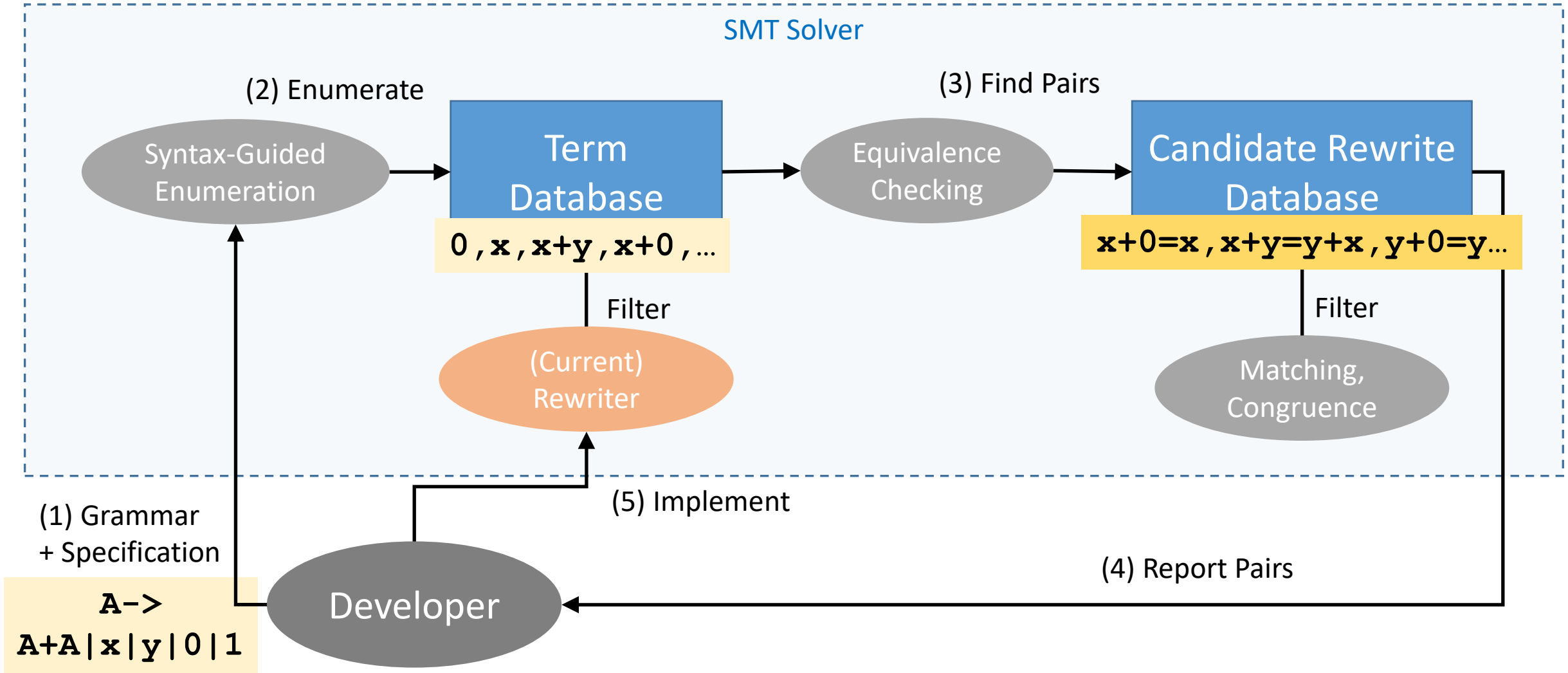


(Partially) Automated Rewrite Rule Generation



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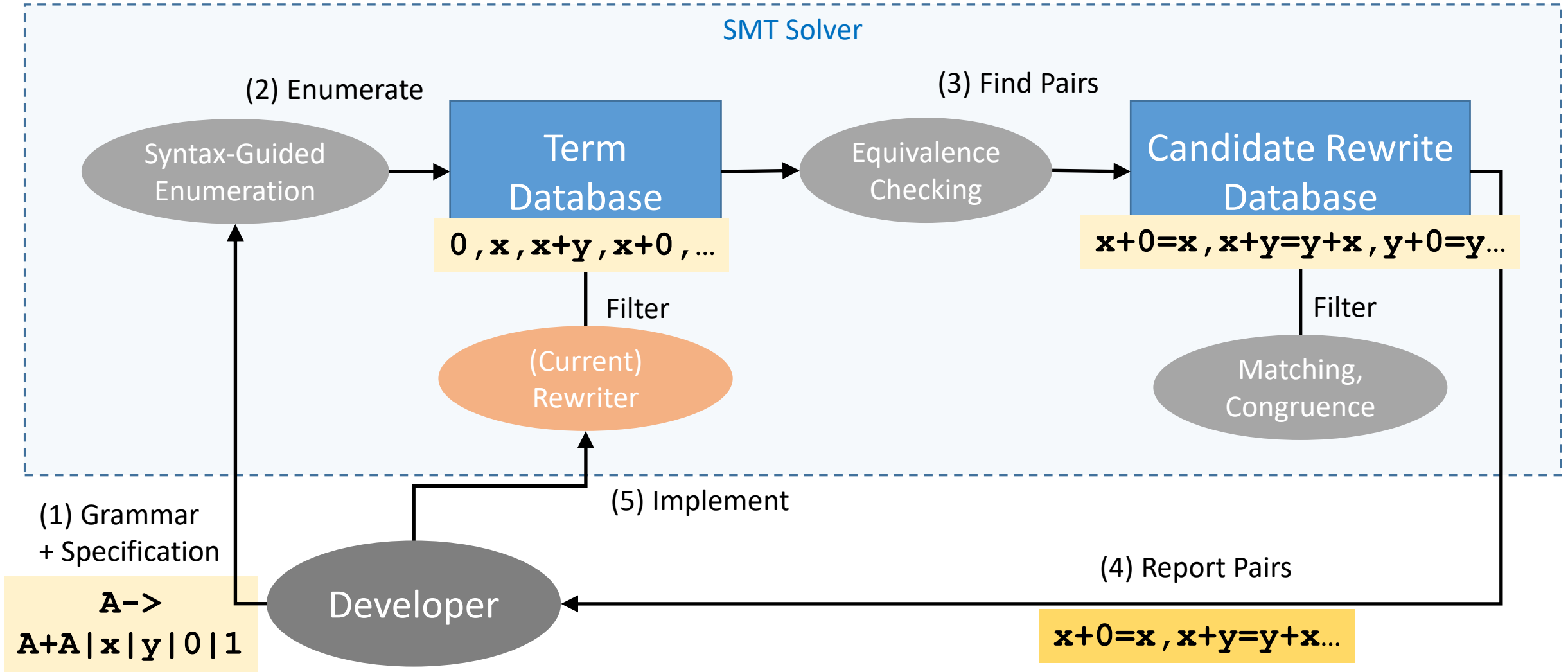

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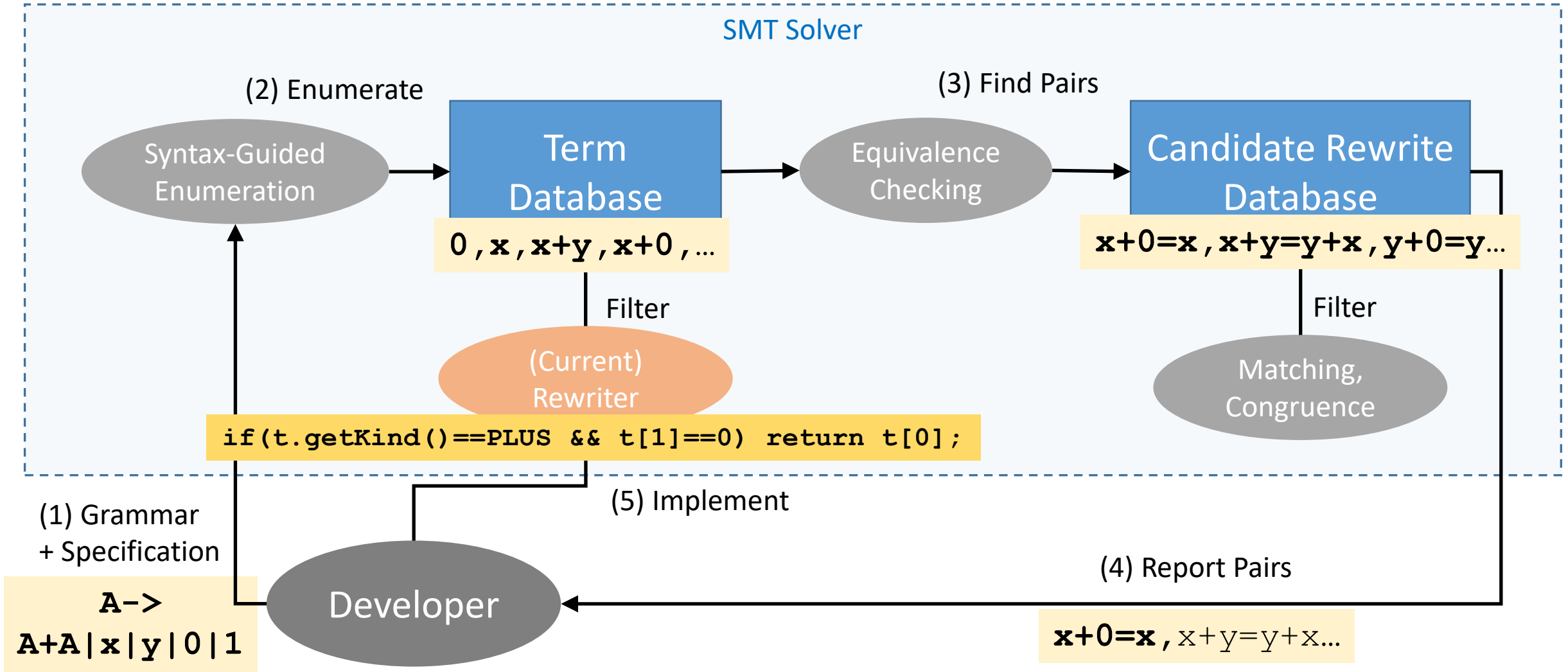
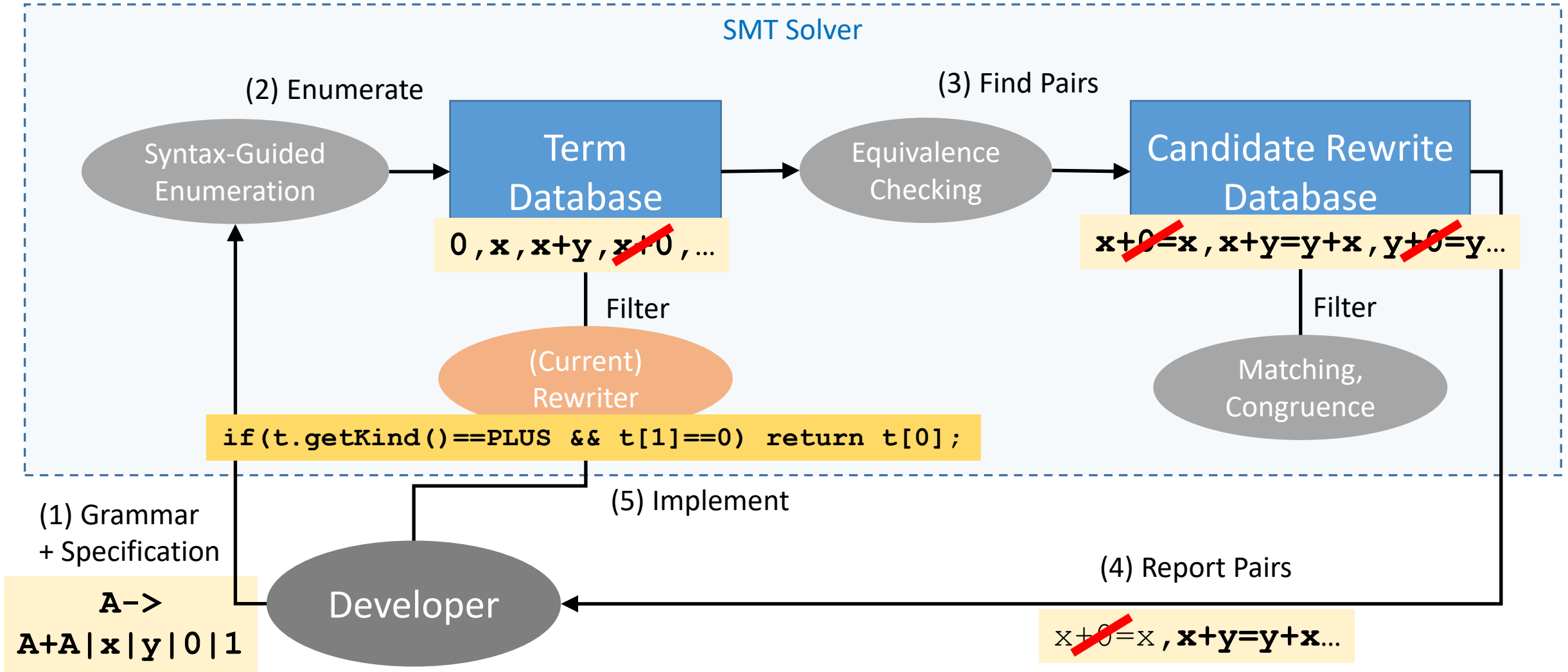
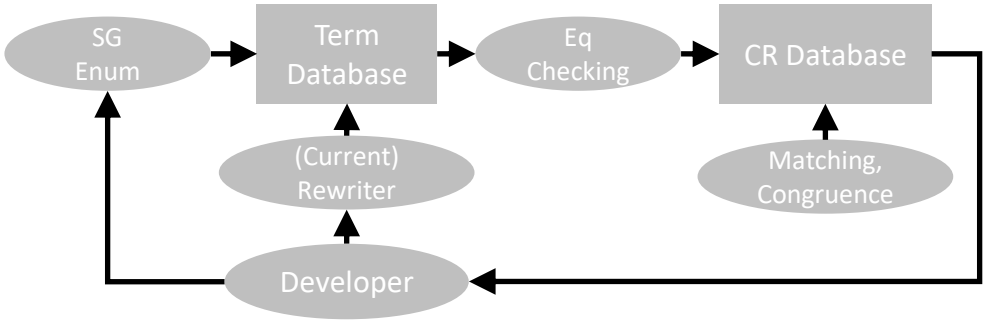


(Partially) Automated Rewrite Rule Generation



Experience

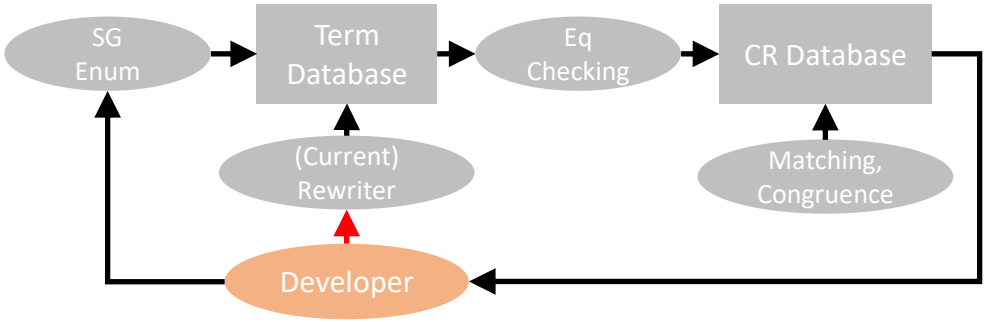



```
(synth-fun f
 ((x String) (y String) (z Int))
 String (
 (Start String (
  x y "A" "B" ""
  (str.++ Start Start)
  (str.replace Start Start Start)
  (str.at Start ie)
  (int.to.str ie)
  (str.substr Start ie ie)))
 (ie Int (
  0 1 z
  (+ ie ie)
  (- ie ie)
  (str.len Start)
  (str.to.int Start)
  (str.indexof Start Start ie)))))
```

```
(synth-fun f ((s (BitVec 4))
              (t (BitVec 4)))
 (BitVec 4) (
 (Start (BitVec 4) (
  s t #x0
  (bvneg  Start)
  (bvnot  Start)
  (bvadd  Start Start)
  (bvmul  Start Start)
  (bvand  Start Start)
  (bvlshr Start Start)
  (bvor   Start Start)
  (bvshl  Start Start)))))
```

```
(synth-fun f
 ((x Bool) (y Bool)
  (z Bool) (w Bool))
 Bool (
 (Start Bool (
  (and d1 d1) (not d1)
  (or d1 d1) (xor d1 d1)))
 (d1 Bool (
  x (and d2 d2) (not d2)
  (or d2 d2) (xor d2 d2)))
 (d2 Bool (
  w (and d3 d3) (not d3)
  (or d3 d3) (xor d3 d3)))
 (d3 Bool (
  y (and d4 d4) (not d4)
  (or d4 d4) (xor d4 d4)))
 (d4 Bool (z))))
```

Examples of Rewrites



- Bit-Vectors

  `bvlshr(x,x)` → `#x0000` `x-(x&y)` → `x&~y` `concat(#x1,x)=concat(#x0,y)` → ⊥

  `x+1` → `~(-x)` `(x&y)+(x|y)` → `x+y` `bvxor(x,x&y)` → `~y&x`

- Strings

  `x++"A"="B"++x` → ⊥ `indexof("ABCDE",x,3)` → `indexof("AAADE",x,3)`

  `contains(x,x++"A")` → ⊥ `replace(x,x++y,y)` → `replace(x,x++y,"")`

  `"A"++x="A"++y` → `x=y` `contains("ABCD","C"++x++"B")` → ⊥

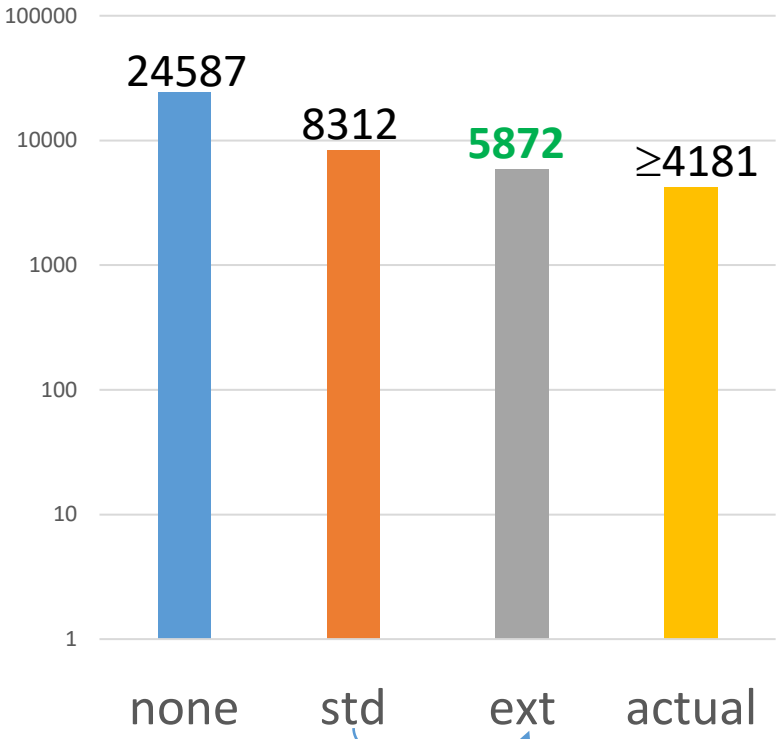
- Booleans

  A∧(A∨B) → A∧B (A∨C)∧(A∨B) → A∧(C∨B)

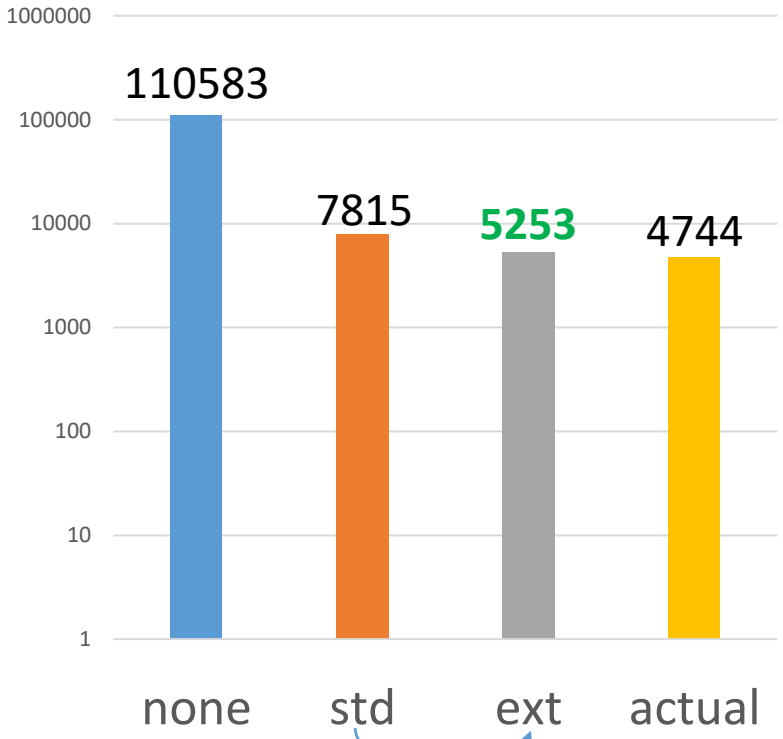
  A=A&B → ¬A∨B (A∨B)=(A∨B∨C) → A∨B∨¬C

Statistics: cvc5's Rewriter (circa 2019):

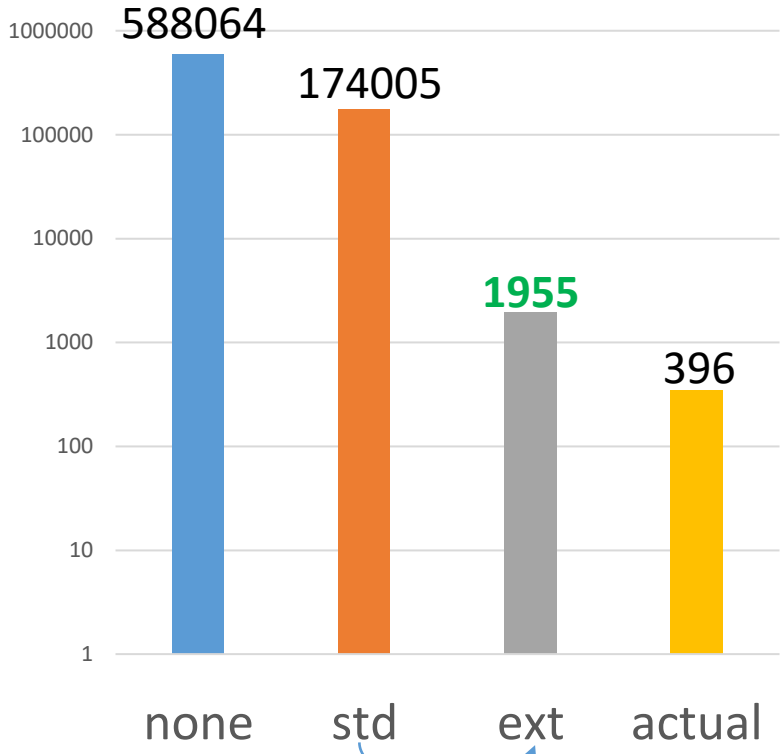
string-term, depth 2



bv-term, depth 3



bool-crci, depth 7



	string-term, depth 2		bv-term, depth 3		bool-crci, depth 7	
%redundant:	49.7%	**28.8%**	39.3%	**9.7%**	99.8%	**82.2%**
time to enumerate:	90.0	**60.8**	96.4	**55.4**	19128.8	**60.6**

[Noetzli/Reynolds/Barbosa/Niemetz/Preiner/Barrett/Tinelli SAT 19]

Impact of Aggressive Simplification *for Strings*

Set		all	-arith	-contain	-msets	Z3	OSTRICH
CMU	sat	7947	7746	**7948**	7946	4585	
	unsat	**66**	31	**66**	**66**	52	
	×	173	409	172	174	3549	
TERMEQ	sat	**10**	**10**	**10**	**10**	1	
	unsat	**49**	36	27	**49**	36	
	×	22	35	44	22	44	
SLOG	sat	**1302**	**1302**	**1302**	**1302**	1100	1289
	unsat	**2082**	**2082**	**2082**	**2082**	2075	**2082**
	×	7	7	7	7	216	20
APLAS	sat	**132**	**132**	**132**	**132**	10	
	unsat	**292**	291	171	171	94	
	×	159	160	280	280	479	
Total	sat	9391	9190	**9392**	9390	5696	1289
	unsat	**2489**	2440	2346	2368	2257	2082
	×	361	611	503	483	4288	8870

[Reynolds/Noetzli/Tinelli/Barrett CAV 19]

-arith: w/o arithmetic simplifications
-contain: w/o contain-based simplifications
-mset: w/o multiset-based simplifications

- cvc5 uses >5000 lines of C++ for simplification rules (and growing)
  - Important aspect of modern string solving

Application #2: Improving Confidence in the SMT Solver

Improving Confidence: Rewriting

- *Does my SMT solver implement any unsound rewrites?*

Improving Confidence: Rewriting

Grammar of interest

```
(synth-fun f ((s BV8) (t BV8)) BV8
   ((Start BV8
      (s t #x00 #x01
       ~Start
       -Start
       bvlshr(Start Start)
       Start & Start
       Start + Start
   )))
```

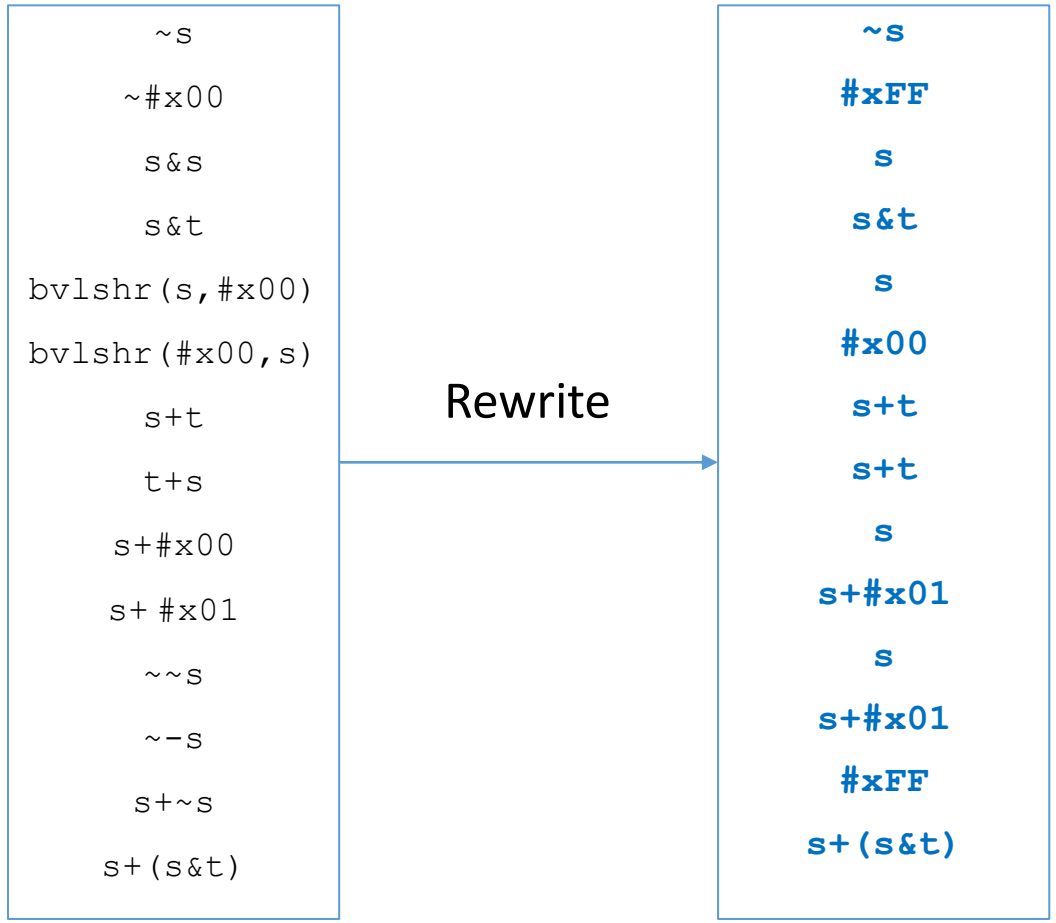
Improving Confidence: Rewriting

```
~s
~#x00
s&s
s&t
bvlshr(s,#x00)
bvlshr(#x00,s)
s+t
t+s
s+#x00
s+#x01
~~s
~-s
s+~s
s+(s&t)
```

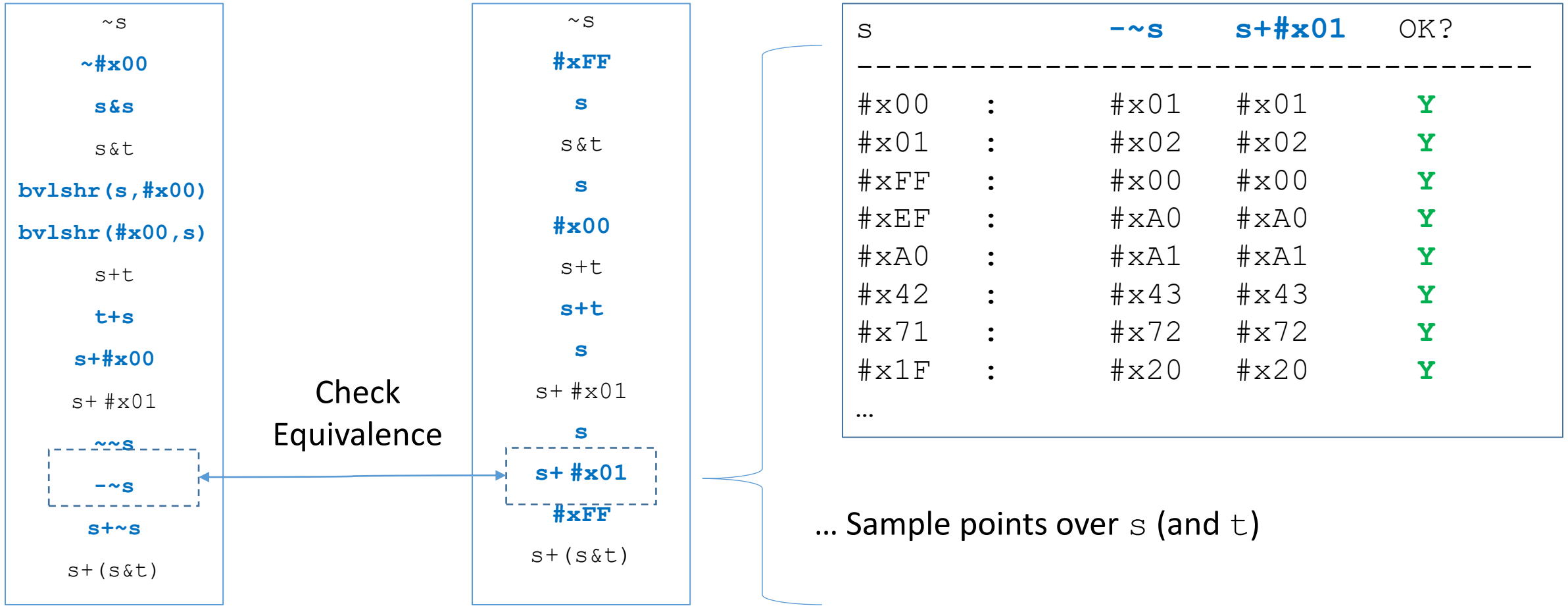
Syntax-Guided Term Enumeration

```
(synth-fun f ((s BV8) (t BV8)) BV8
   ((Start BV8
      (s t #x00 #x01
       ~Start
       -Start
       bvlshr(Start Start)
       Start & Start
       Start + Start
   )))
```

Improving Confidence: Rewriting



Improving Confidence: Rewriting

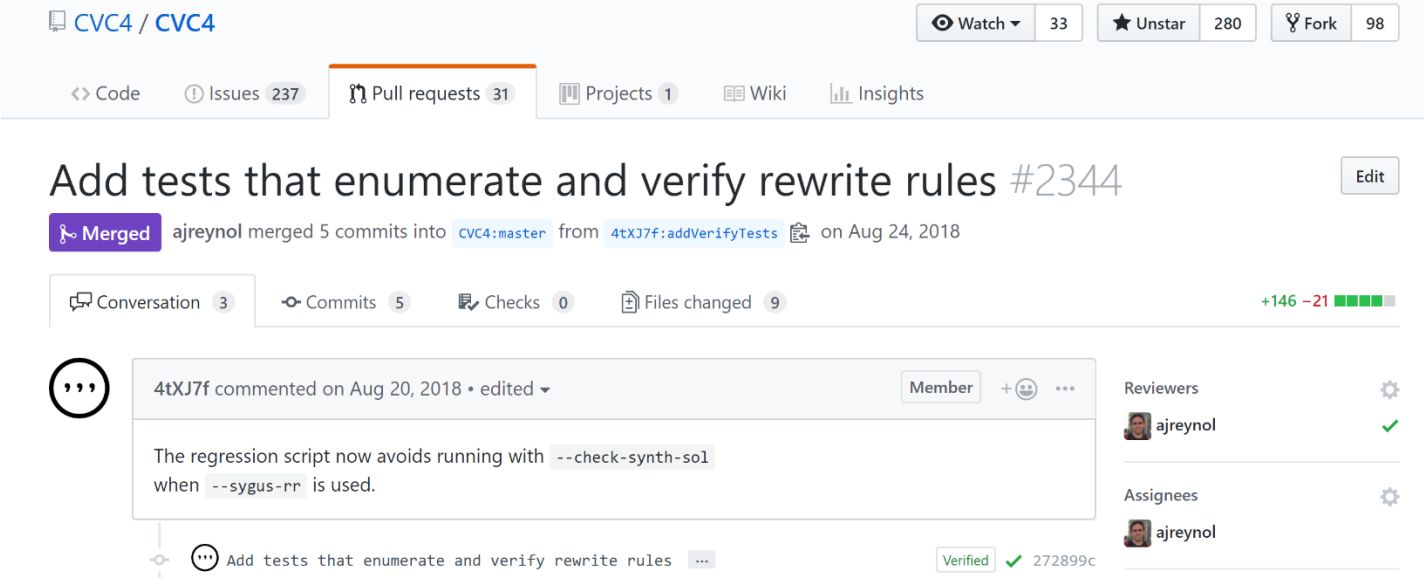


Improving Confidence: Rewriting

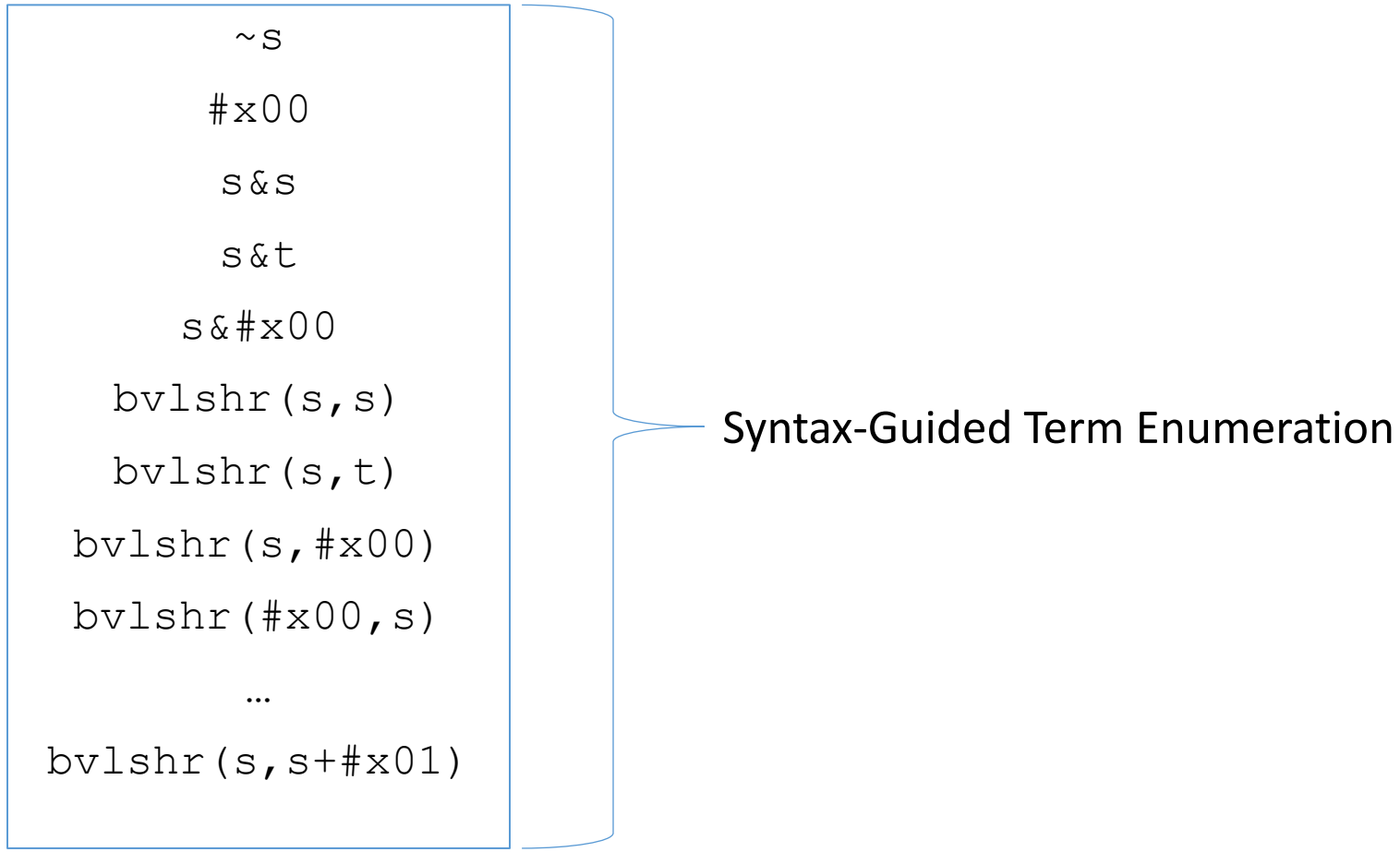
⇒Has been critical for *finding bugs* in newly written rewriter code

```
(unsound-rewrite (bvuge (bvadd x #x0001) x) true)
; --sygus-rr-verify detected unsoundness in the rewriter!
; Terms have the same rewritten form but are not equivalent
; for x=#xFFFF, where they evaluate to:
; (bvuge (bvadd x #x0001) x) = false
; true = true
```

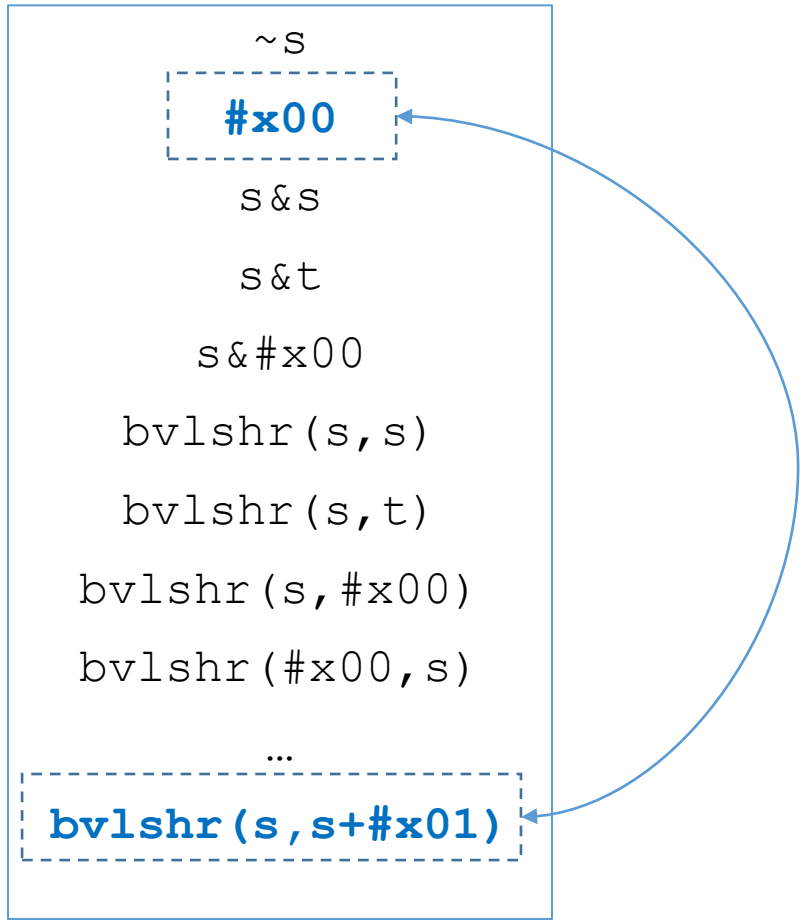
⇒Run as part of cvc5's regression tests:



Improving Confidence: Query Generation



Improving Confidence: Query Generation

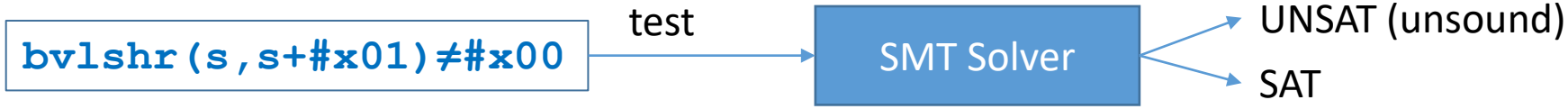


Check Equivalence

s		#x00	bvlshr(s,s+#x01)	=?
#x00	:	#x00	#x01	Y
#x01	:	#x00	#x00	Y
#xFF	:	#x00	#xFF	N
#xEF	:	#x00	#x00	Y
#xA0	:	#x00	#x00	Y
#x42	:	#x00	#x00	Y
#x71	:	#x00	#x00	Y
#x1F	:	#x00	#x00	Y

…

- **Intuition:** terms that are “almost equivalent” => inputs that are “almost unsatisfiable”



Application #3: Solving Quantified Inputs via Invertibility Conditions

Solving Quantified Bit-Vectors: Example

- Consider quantifier elimination problem for bit-vectors:

$$\exists x{:}BV_8.\, s*x=t \;\Leftrightarrow\; ???$$

$\Rightarrow$ Applications in proving safety properties, compiler optimzations

- Common solving technique is model-based instantiation **[Wintersteiger et al 2013]**
  - Based on *values*: above is equivalent $s*\#x00=t \vee s*\#x01=t \vee s*\#x02=t \vee \ldots$
  - Scalability issues for larger bit-widths

- **Idea:** Use SyGuS to find *invertibility conditions* for bit-vector equations

  $\Rightarrow$ Compact form for quantifier elimination

What is an Invertibility Condition?

- Some equations are “invertible”, e.g.:
  - `s+x=t` … always has solution **`x=t-s`**
- Some equations are not, e.g.:
  - `s*x=t` … `x` has *no solution* when, e.g. `s=2`, `t=3`

[Niemetz/Preiner/Reynolds/Barrett/Tinelli CAV 2018]

What is an Invertibility Condition?

- `s+x=t` … always has solution **`x=t-s`**
- `s*x=t` … `x` has *no solution* when, e.g. `s=2`, `t=3`

- **Challenge:** it is possible to characterize exactly when `x` has a solution?
  - Find a predicate `IC(s,t)` such that `x` has a solution iff `IC(s,t)` is satisfiable
  - We call `IC` an "invertibility condition"
  - Invertibility Conditions $\Rightarrow$ QE procedure for "linear" formulas

[Niemetz/Preiner/Reynolds/Barrett/Tinelli CAV 2018]

What is an Invertibility Condition?

- For terms `s`, `t`, operator $\oplus$ and relation ~
  - An *invertibility condition* `IC(s,t)` is a quantifier-free formula such that:

$$\exists x.(x \oplus s \sim t) \Leftrightarrow IC(s,t)$$

    is T-valid.

Finding Invertibility Conditions for BV

$$\exists x.(x \oplus s \sim t) \Leftrightarrow IC(s,t)$$

- Applied to theory of Bit-Vectors **[Niemetz et al CAV 2018]**
- Signature of BV has 15 operators, 4 relations (=/≠, signed/unsigned inequality)
  $\Rightarrow$ Total of 162 invertibility conditions to find
- Some took several hours to find by hand $\therefore$ *use SyGuS*
  - Using SyGuS, found **118** of **162** conditions
    - Many simpler than hand-crafted ones
  - When combined with hand-crafted ICs, found all **162** conditions

Invertibility Conditions for BV

$\ell[x]$	$\approx$	$\not\approx$
$x \cdot s \bowtie t$	$(-s \mid s)\ \&\ t \approx t$	$s \not\approx 0 \vee t \not\approx 0$
$x \bmod s \bowtie t$	$\sim(-s) \geq_u t$	$s \not\approx 1 \vee t \not\approx 0$
$s \bmod x \bowtie t$	$(t + t - s)\ \&\ s \geq_u t$	$s \not\approx 0 \vee t \not\approx 0$
$x \div s \bowtie t$	$(s \cdot t) \div s \approx t$	$s \not\approx 0 \vee t \not\approx \sim 0$
$s \div x \bowtie t$	$s \div (s \div t) \approx t$	$\begin{cases} s\ \&\ t \approx 0 & \text{for } \kappa(s) = 1 \\ \top & \text{otherwise} \end{cases}$
$x\ \&\ s \bowtie t$	$t\ \&\ s \approx t$	$s \not\approx 0 \vee t \not\approx 0$
$x \mid s \bowtie t$	$t \mid s \approx t$	$s \not\approx \sim 0 \vee t \not\approx \sim 0$
$x \gg s \bowtie t$	$(t \ll s) \gg s \approx t$	$t \not\approx 0 \vee s <_u \kappa(s)$
$s \gg x \bowtie t$	$\bigvee_{i=0}^{\kappa(s)} s \gg i \approx t$	$s \not\approx 0 \vee t \not\approx 0$
$x \gg_a s \bowtie t$	$(s <_u \kappa(s) \Rightarrow (t \ll s) \gg_a s \approx t) \wedge$ $(s \geq_u \kappa(s) \Rightarrow (t \approx \sim 0 \vee t \approx 0))$	$\top$
$s \gg_a x \bowtie t$	$\bigvee_{i=0}^{\kappa(s)} s \gg_a i \approx t$	$(t \not\approx 0 \vee s \not\approx 0) \wedge$ $(t \not\approx \sim 0 \vee s \not\approx \sim 0)$
$x \ll s \bowtie t$	$(t \gg s) \ll s \approx t$	$t \not\approx 0 \vee s <_u \kappa(s)$
$s \ll x \bowtie t$	$\bigvee_{i=0}^{\kappa(s)} s \ll i \approx t$	$s \not\approx 0 \vee t \not\approx 0$
$x \circ s \bowtie t$	$s \approx t[\kappa(s) - 1 : 0]$	$\top$
$s \circ x \bowtie t$	$s \approx t[\kappa(t) - 1 : \kappa(t) - \kappa(s)]$	$\top$

Table 2. Conditions for the invertibility of bit-vector operators over (dis)equality. Those for $\cdot$, $\&$ and $\mid$ are given modulo commutativity of those operators.

ICs for =,≠

…not pictured:
ICs for `bvuge`, `bvugt`, `bvsge`, `bvsgt`

[Niemetz/Preiner/Reynolds/Barrett/Tinelli CAV 2018]

Invertibility Conditions for BV

$\ell[x]$	$\approx$	$\not\approx$
$x \cdot s \bowtie t$	$(-s \mid s)\ \&\ t \approx t$	$s \not\approx 0 \vee t \not\approx 0$
$x \bmod s \bowtie t$	$\sim(-s) \geq_u t$	$s \not\approx 1 \vee t \not\approx 0$
$s \bmod x \bowtie t$	$(t + t - s)\ \&\ s \geq_u t$	$s \not\approx 0 \vee t \not\approx 0$
$x \div s \bowtie t$	$(s \cdot t) \div s \approx t$	$s \not\approx 0 \vee t \not\approx \sim 0$
$s \div x \bowtie t$	$s \div (s \div t) \approx t$	$\begin{cases} s\ \&\ t \approx 0 & \text{for } \kappa(s) = 1 \\ \top & \text{otherwise} \end{cases}$
$x\ \&\ s \bowtie t$	$t\ \&\ s \approx t$	$s \not\approx 0 \vee t \not\approx 0$
$x \mid s \bowtie t$	$t \mid s \approx t$	$s \not\approx \sim 0 \vee t \not\approx \sim 0$
$x \gg s \bowtie t$	$(t \ll s) \gg s \approx t$	$t \not\approx 0 \vee s <_u \kappa(s)$
$s \gg x \bowtie t$	$\bigvee_{i=0}^{\kappa(s)} s \gg i \approx t$	$s \not\approx 0 \vee t \not\approx 0$
$x \gg_a s \bowtie t$	$(s <_u \kappa(s) \Rightarrow (t \ll s) \gg_a s \approx t) \wedge$ $(s \geq_u \kappa(s) \Rightarrow (t \approx \sim 0 \vee t \approx 0))$	$\top$
$s \gg_a x \bowtie t$	$\bigvee_{i=0}^{\kappa(s)} s \gg_a i \approx t$	$(t \not\approx 0 \vee s \not\approx 0) \wedge$ $(t \not\approx \sim 0 \vee s \not\approx \sim 0)$
$x \ll s \bowtie t$	$(t \gg s) \ll s \approx t$	$t \not\approx 0 \vee s <_u \kappa(s)$
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$x \circ s \bowtie t$	$s \approx t[\kappa(s) - 1 : 0]$	$\top$
$s \circ x \bowtie t$	$s \approx t[\kappa(t) - 1 : \kappa(t) - \kappa(s)]$	$\top$

Table 2. Conditions for the invertibility of bit-vector operators over (dis)equality. Those for $\cdot$, $\&$ and $\mid$ are given modulo commutativity of those operators.

$$\exists x.\ s * x = t$$
$$\Leftrightarrow$$
$$(-s \mid s)\ \&\ t = t$$

- Condition above encodes
  "s has fewer trailing zeroes than t"
- Conditions like this one are concise, subtle
  $\Rightarrow$ Excellent target for SyGuS

[Niemetz/Preiner/Reynolds/Barrett/Tinelli CAV 2018]

Experimental Results

- Quantifier Instantiation based on Invertibility Conditions in CVC4
- Won quantified bit-vector (BV) category of SMT-COMP 2018

unsat	Boolector	CVC4	Q3B	Z3	cegqi$_m$	cegqi$_k$	cegqi$_s$	cegqi$_b$
h-uauto	14	12	93	24	10	103	105	**106**
keymaera	3917	3790	3781	**3923**	3803	3798	3888	3918
psyco	**62**	**62**	49	**62**	**62**	39	**62**	61
scholl	57	36	13	**67**	36	27	36	35
tptp	55	52	**56**	**56**	**56**	**56**	**56**	**56**
uauto	**137**	72	131	**137**	72	72	135	**137**
ws-fixpoint	74	71	**75**	74	**75**	74	**75**	**75**
ws-ranking	16	8	18	**19**	15	11	12	11
Total unsat	4332	4103	4216	4362	4129	4180	4369	**4399**

sat	Boolector	CVC4	Q3B	Z3	cegqi$_m$	cegqi$_k$	cegqi$_s$	cegqi$_b$
h-uauto	15	10	**17**	13	16	**17**	16	**17**
keymaera	**108**	21	24	**108**	20	13	36	75
psyco	131	**132**	50	131	**132**	60	**132**	129
scholl	**232**	160	201	204	203	188	208	211
tptp	**17**	**17**	**17**	**17**	**17**	**17**	**17**	**17**
uauto	14	14	15	**16**	14	14	14	14
ws-fixpoint	45	49	**54**	36	45	51	49	50
ws-ranking	19	15	**37**	33	33	31	31	32
Total sat	**581**	418	415	558	480	391	503	545

Total (5151)	4913	4521	4631	4920	4609	4571	4872	**4944**

Table 4. Results on satisfiable and unsatisfiable benchmarks with a 300 second timeout.

Invertibility Conditions for Floating Points

- Say I want to find invertibility condition for *floating point* equation:

$$\exists x : FP_{e,s} . x \oplus s \sim t$$

- Can we apply the same approach to Floating Points?
  - …will it scale?

Invertibility Conditions for Floating Points

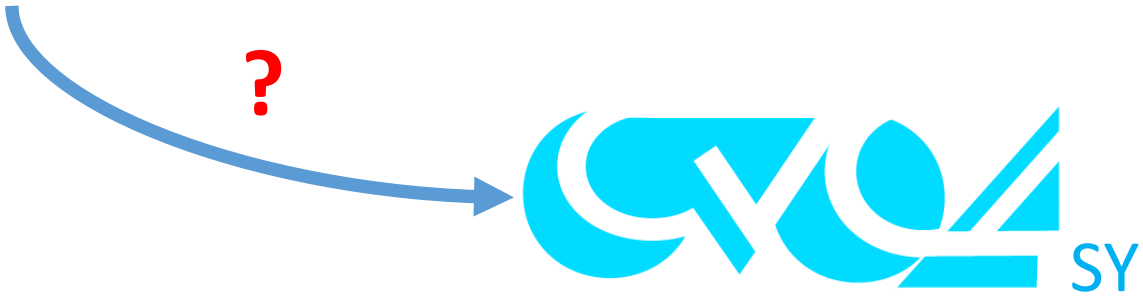
$$\exists \mathbf{IC}.\ \forall st{:}FP_{e,s}.\ (\exists x{:}FP_{e,s}.x \oplus s \sim t) \Leftrightarrow \mathbf{IC}(s,t)$$

- Can express as a synthesis problem:

  …there exists a predicate **IC** that is equivalent to $\exists x{:}FP_{e,s}.x \oplus s \sim t$

Invertibility Conditions for Floating Points

$$\exists IC.\ \forall st{:}FP_{e,s}.\ (\exists x{:}FP_{e,s}.x \oplus s \sim t) \Leftrightarrow IC(s,t)$$



… solve via syntax-guided synthesis (SyGuS)?

How do we find Invertibility Conditions?

$$\exists IC.\ \forall st{:}FP_{\mathbf{e},\mathbf{s}}.\ (\exists x{:}FP_{\mathbf{e},\mathbf{s}}.x \oplus s \sim t) \Leftrightarrow IC(s,t)$$

- **Challenge #1:** problem is parameteric
  - Parameterized by # exponent bits **e**, # significant bits **s**

How do we find Invertibility Conditions?

$$\exists IC.\ \forall st{:}FP_{3,5}.\ (\exists x{:}FP_{3,5}.x \oplus s \sim t) \Leftrightarrow IC(s,t)$$

- **Challenge #1:** problem is parameteric

$\Rightarrow$ Solve for a *fixed format*, check if solution generalizes **[Niemetz et al CAV 2018]**

- Choose e=**3**, s=**5** … large enough to exhibit FP behaviors, small enough for synthesis to scale

How do we find Invertibility Conditions?

$$\exists IC.\ \forall st{:}FP_{3,5}.\ (\exists x{:}FP_{3,5}.x \oplus s \sim t) \Leftrightarrow IC(s,t)$$

How do we find Invertibility Conditions?

$$\exists IC.\ \forall st{:}FP_{3,5}.\ (\exists x{:}FP_{3,5}.x \oplus s \sim t) \Leftrightarrow IC(s,t)$$

- **Challenge #2:** problem has *three levels* of quantifier alternation
  - …SyGuS solvers only typically handle two

How do we find Invertibility Conditions?

$$\exists IC.\ \forall st{:}FP_{3,5}.\ \Big(\bigvee_{c=1}^{227} c \oplus s \sim t \Big) \Leftrightarrow IC(s,t)$$

- **Challenge #2:** problem has three levels of quantifier alternation

$\Rightarrow$ Expand the innermost quantifier **[Niemetz et al CAV 2018]**

- Since domain is (relatively) small and finite
  - Only 227 values for $FP_{3,5}$

How do we find Invertibility Conditions?

$$\exists IC.\ \forall st{:}FP_{3,5}.\ \Big(\bigvee_{c=1}^{227} c \oplus s \sim t \Big) \Leftrightarrow IC(s,t)$$

…can now give as input to SyGuS solver

How do we find Invertibility Conditions?

$$\exists IC.\ \forall st{:}FP_{3,5}.\ \left(\bigvee_{c=1}^{227} c \oplus s \sim t \right) \Leftrightarrow IC(s,t)$$

- **Challenge #3:** Scalability!
  - With respect to BV synthesis @ 4 bits **[Niemetz et al CAV 2018]**:
    - Roughly 14 times more constraints, q.f. FP is much harder than BV

How do we find Invertibility Conditions?

$$\exists IC.\ \forall st{:}FP_{3,5}.\ \Big(\bigvee_{c=1}^{227} c \oplus s \sim t \Big) \Leftrightarrow IC(s,t)$$

- **Challenge #3:** Scalability

How do we find Invertibility Conditions?

$$\exists IC.\ \bigwedge_{s=1}^{227} \bigwedge_{t=1}^{227} c_{s,t} \Leftrightarrow IC(s,t)$$

$\exists x.x*s=t?$ where s, t are constants

- **Challenge #3:** Scalability

  $\Rightarrow$Statically compute the I/O behavior of *entire domain* of $IC(s,t)$

  - Each $c_{s,t}$ is determined by a quantifier-free satisfiability query
    - 227*227 = 51529, takes ~10 minutes

How do we find Invertibility Conditions?

$$\exists IC. \quad \bigwedge_{s=1}^{227} \bigwedge_{t=1}^{227} c_{s,t} \Leftrightarrow IC(s,t)$$

How do we find Invertibility Conditions?

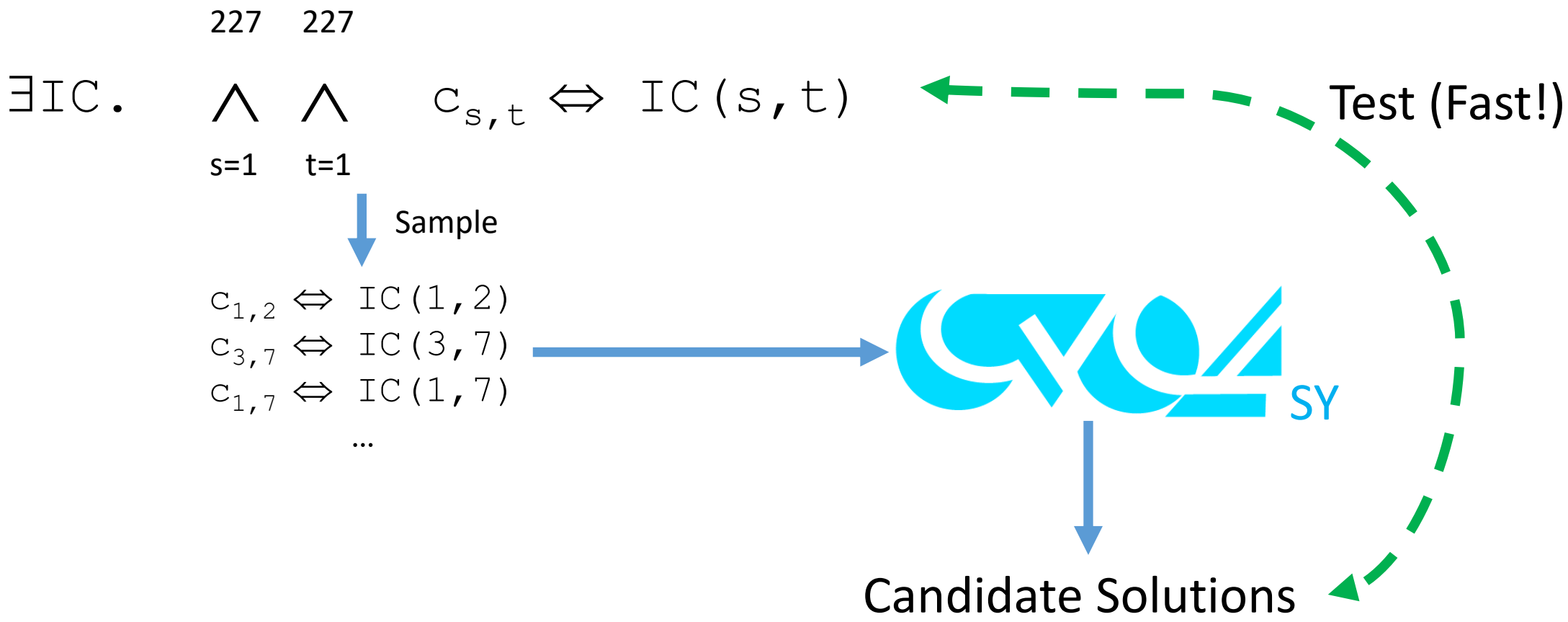
$$\exists IC. \quad \bigwedge_{s=1}^{227} \bigwedge_{t=1}^{227} c_{s,t} \Leftrightarrow IC(s,t)$$

Sample

$c_{1,2} \Leftrightarrow IC(1,2)$
$c_{3,7} \Leftrightarrow IC(3,7)$
$c_{1,7} \Leftrightarrow IC(1,7)$
…

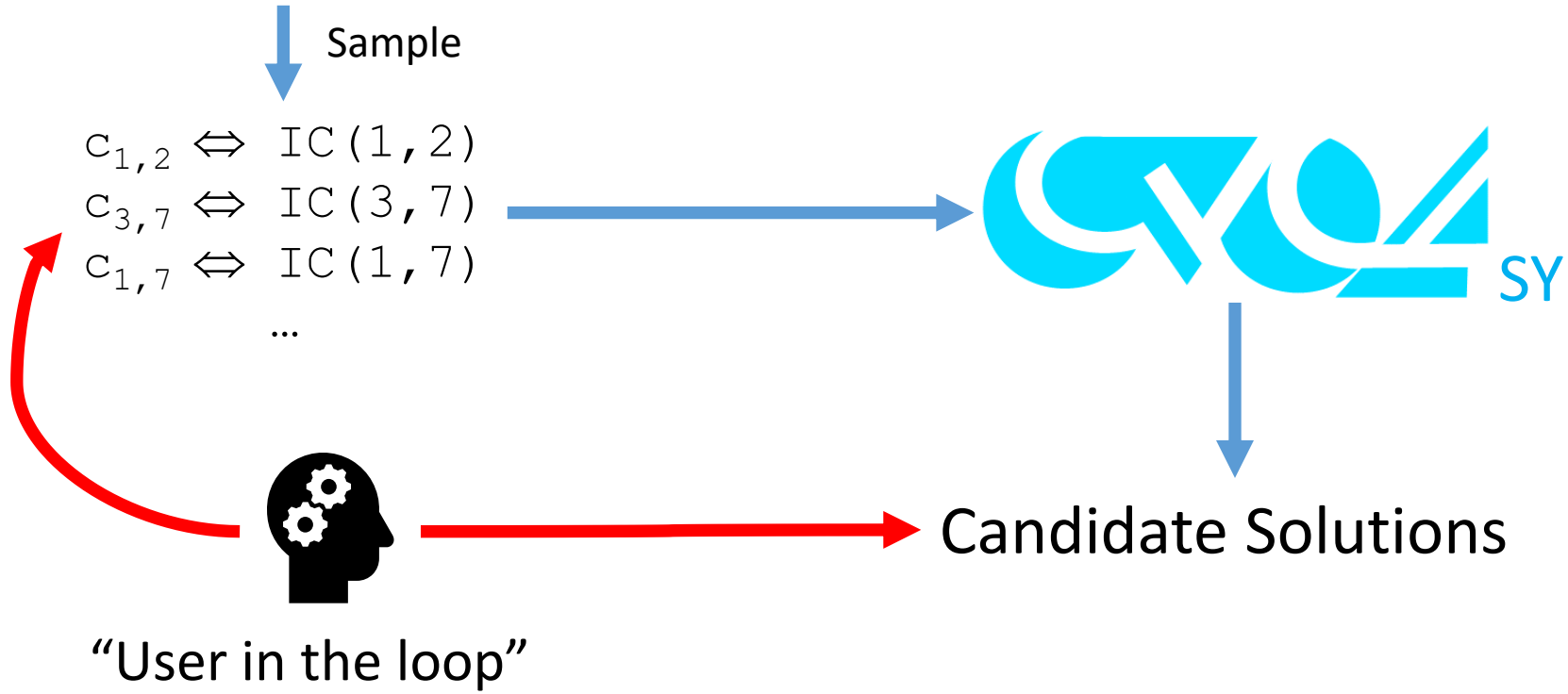
Candidate Solutions

How do we find Invertibility Conditions?



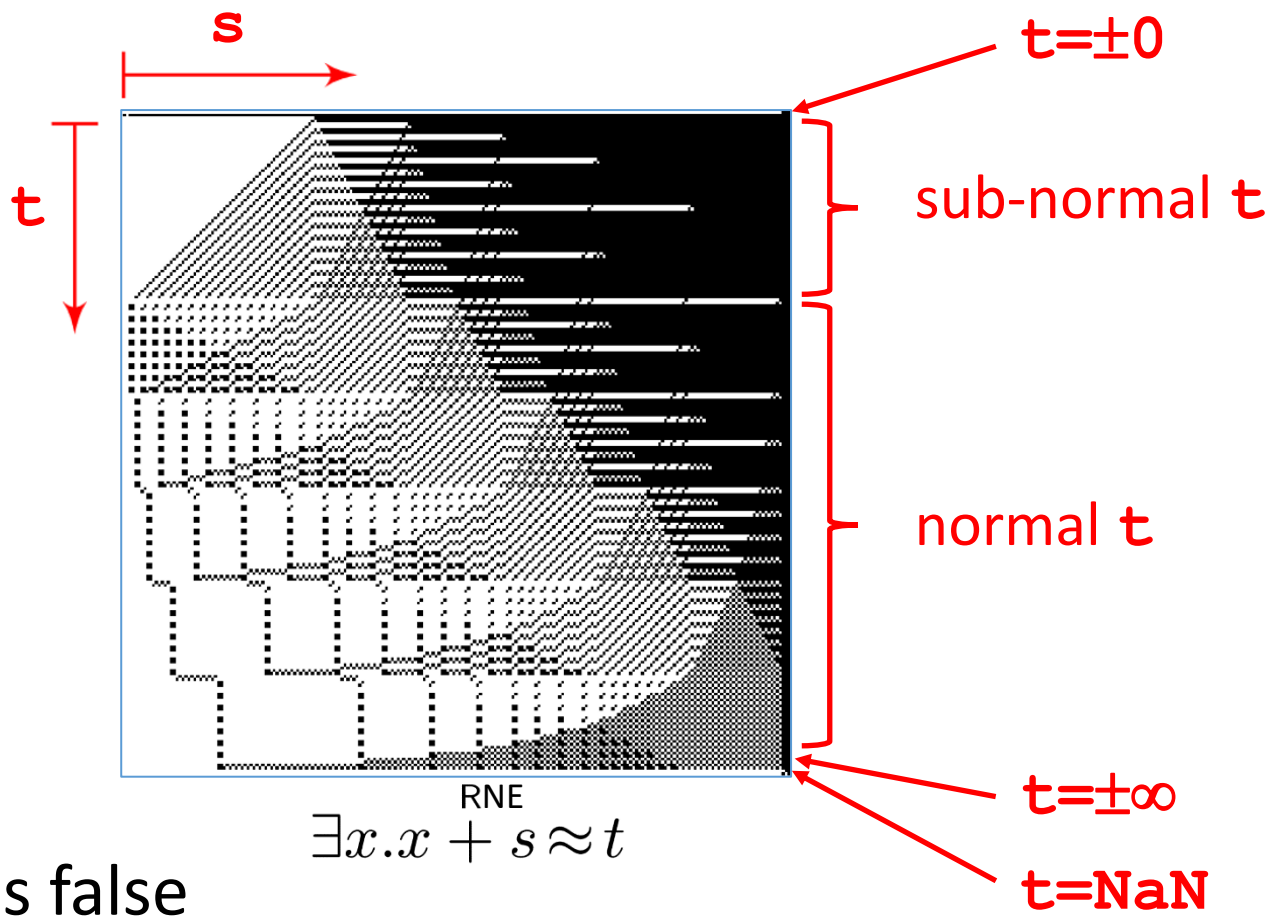
How do we find Invertibility Conditions?

$$\exists IC.\ \bigwedge_{s=1}^{227} \bigwedge_{t=1}^{227} c_{s,t} \Leftrightarrow IC(s,t)$$



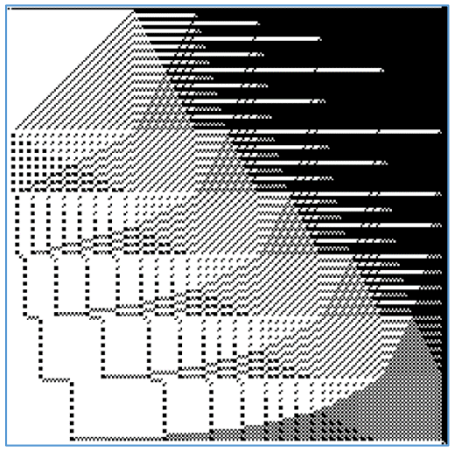
Visualizing I/O Specs of Invertibility Conditions

$$\exists \mathtt{IC}.\ \bigwedge_{s=1}^{227} \bigwedge_{t=1}^{227} c_{s,t} \Leftrightarrow \mathtt{IC}(s,t)$$



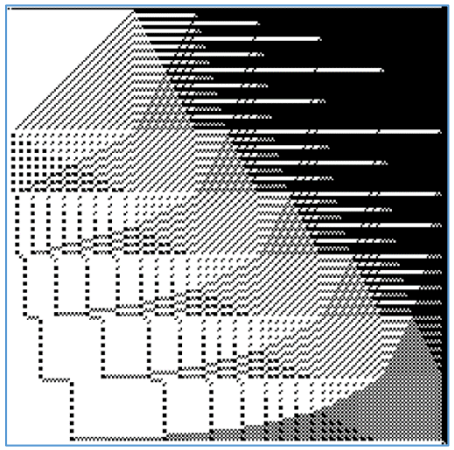
- White = IC is true, Black = IC is false
  - ⇒ Behavior of IC is highly complex!

Synthesizing Invertibility Conditions



SY

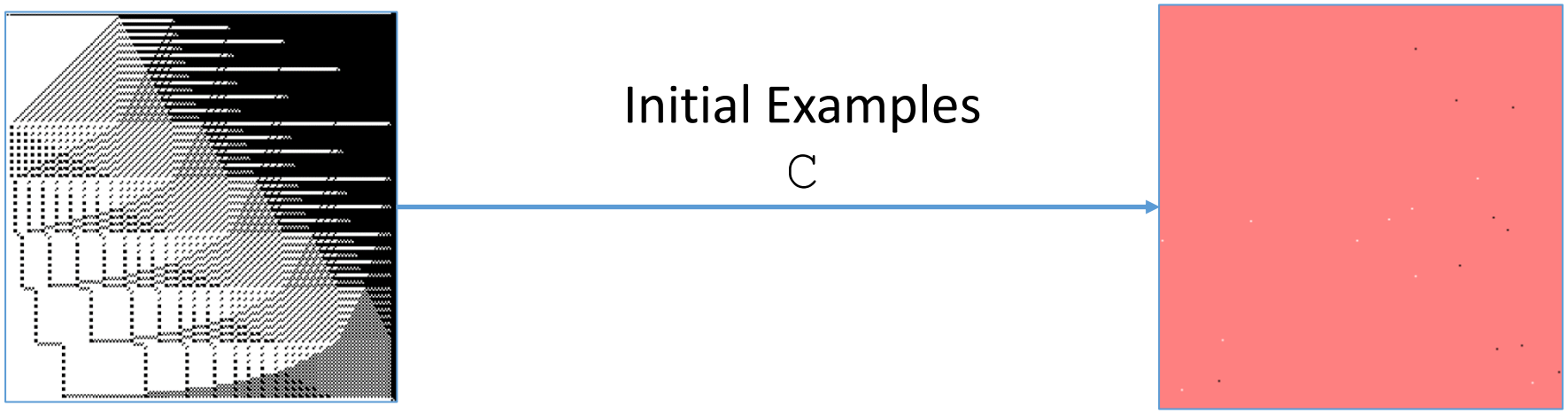
Synthesizing Invertibility Conditions



"Full I/O Specification for IC"
(Verifier)

(Synthesizer)

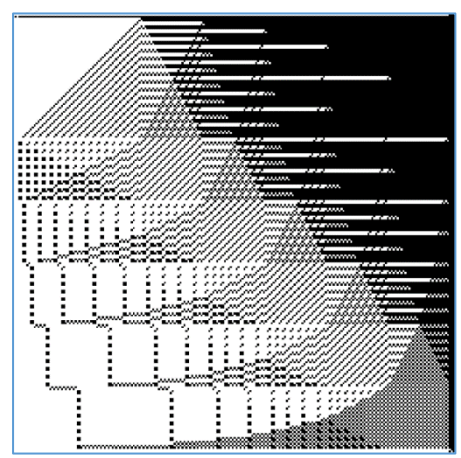
Synthesizing Invertibility Conditions

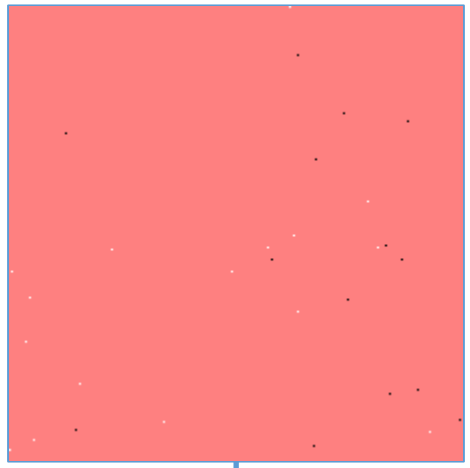


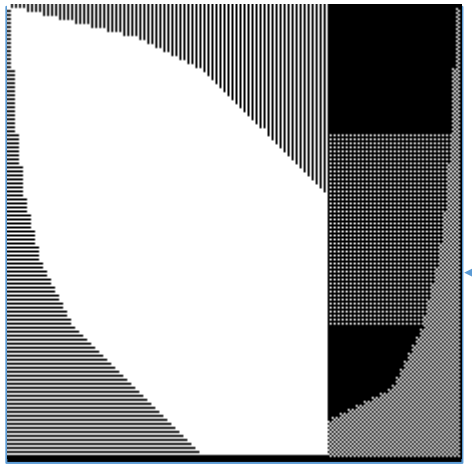
Red = Counterexample not specified

SY

Synthesizing Invertibility Conditions

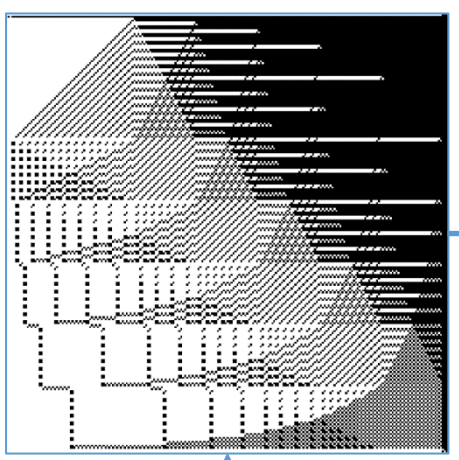






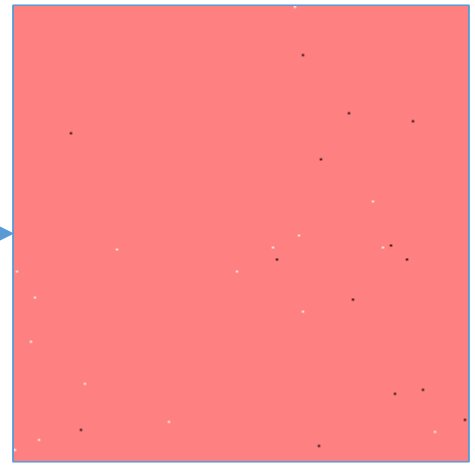
$IC_1(s,t)$

Synthesizing Invertibility Conditions

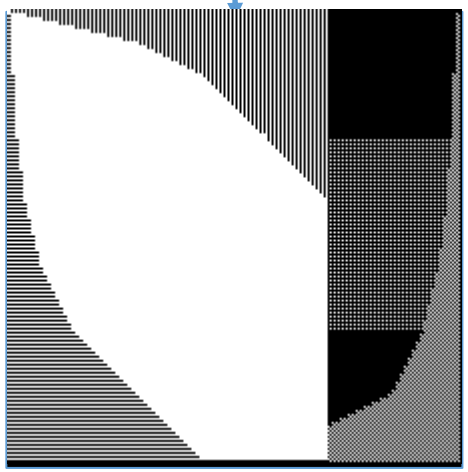


Counterexamples

C, C_1

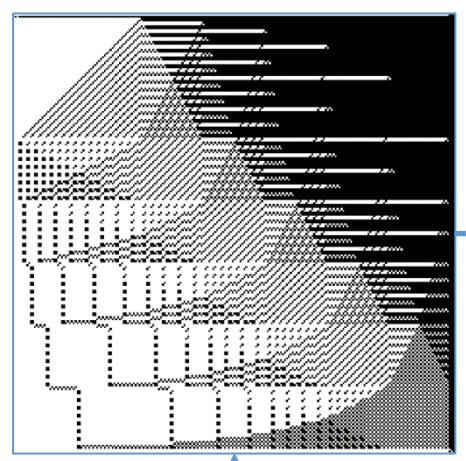


Test



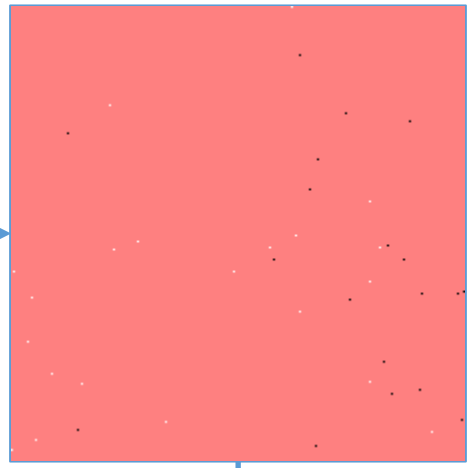
SY

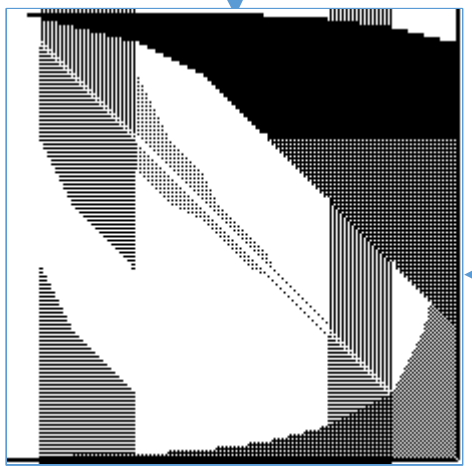
Synthesizing Invertibility Conditions



Counterexamples

C, C_1, C_2

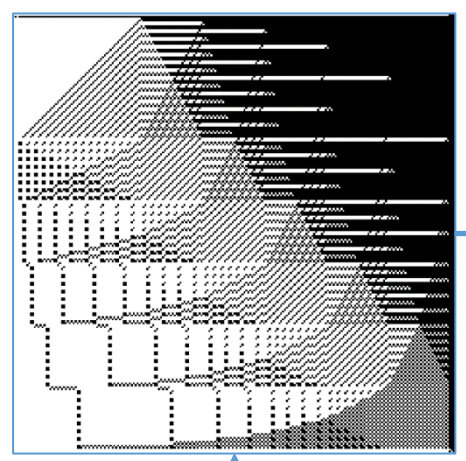




$IC_2(s,t)$

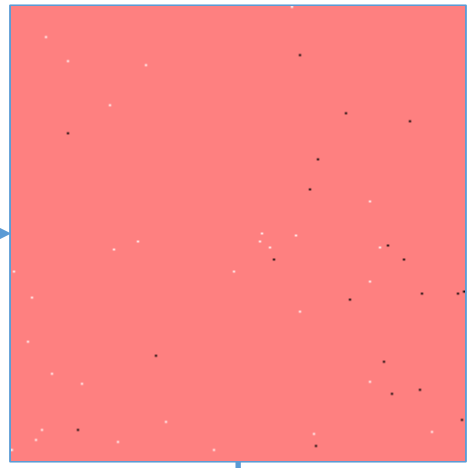
SY

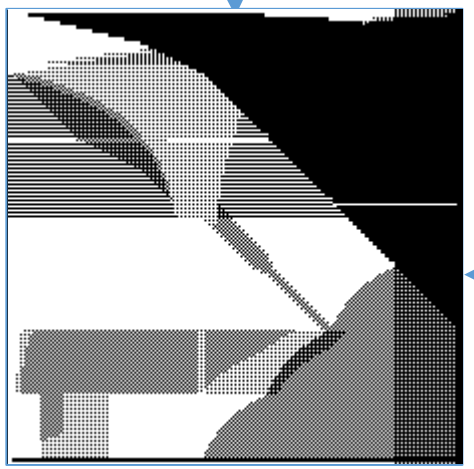
Synthesizing Invertibility Conditions



Counterexamples

C, C_1, C_2, C_3

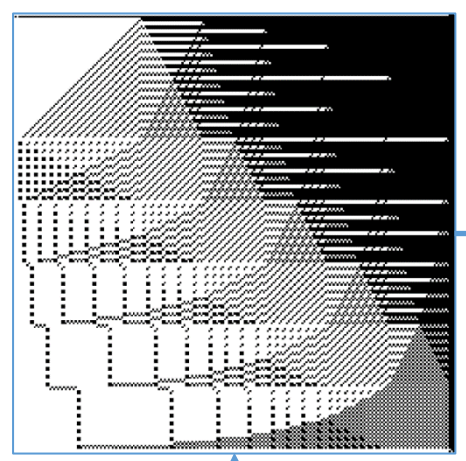




$IC_3(s,t)$

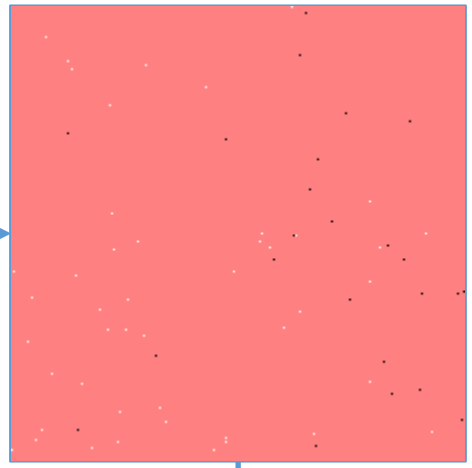
SY

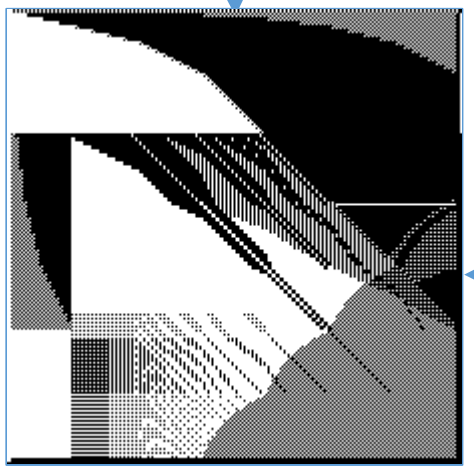
Synthesizing Invertibility Conditions



Counterexamples

C, C_1, C_2, C_3, C_4

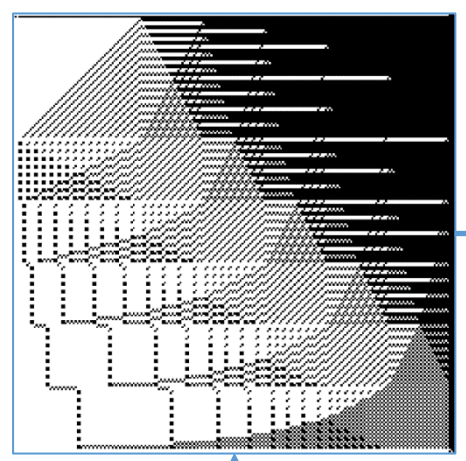




$IC_4(s,t)$

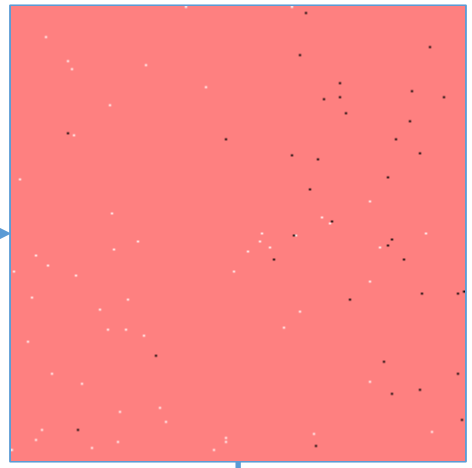
SY

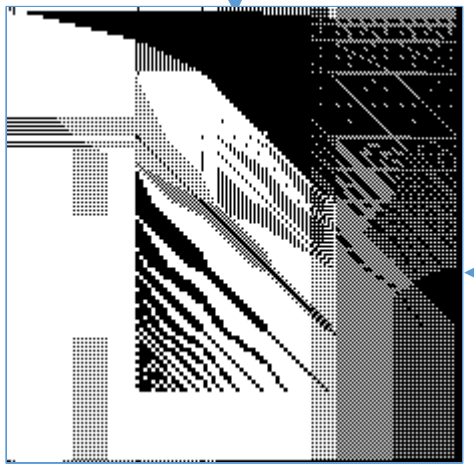
Synthesizing Invertibility Conditions



Counterexamples

$C, C_1, C_2, C_3, C_4, C_5$

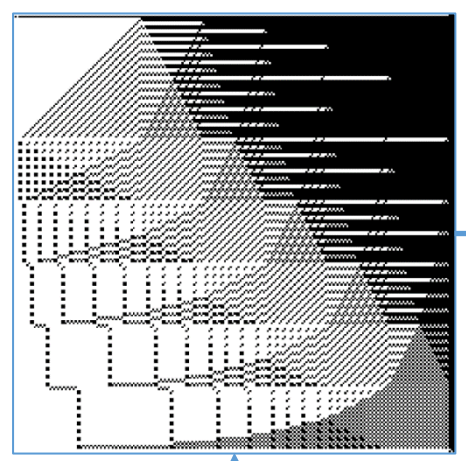




$IC_5(s,t)$

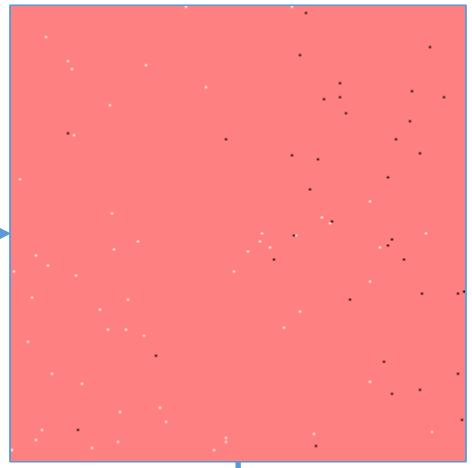
SY

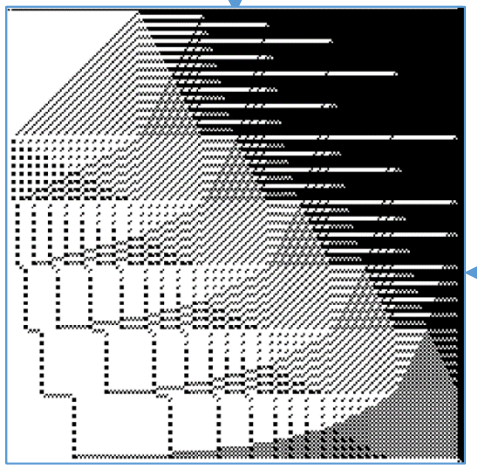
Synthesizing Invertibility Conditions



Counterexamples

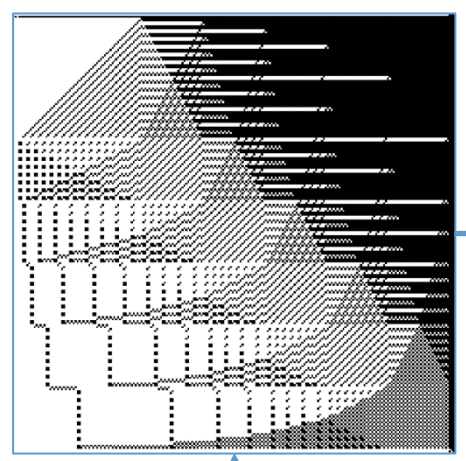
$C, C_1, C_2, C_3, C_4, C_5, C_6$





$$t \approx (t \overset{\text{RTP}}{-} s) \overset{R}{+} s \vee t \approx (t \overset{\text{RTN}}{-} s) \overset{R}{+} s \vee s \approx t$$

Synthesizing Invertibility Conditions



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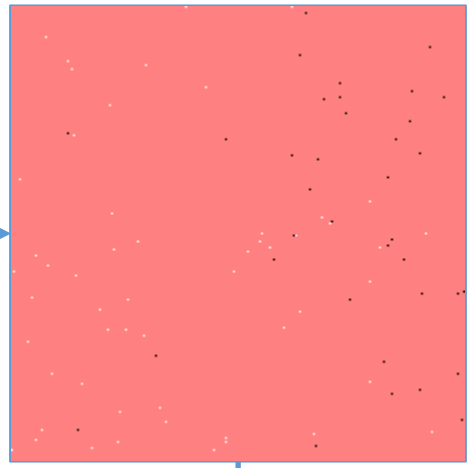

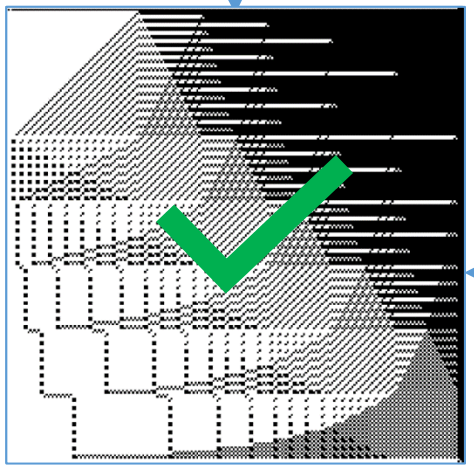

$$t \approx \overset{\text{RTP}}{(t-s)} \overset{R}{+} s \vee t \approx \overset{\text{RTN}}{(t-s)} \overset{R}{+} s \vee s \approx t$$

Finding Invertibility Conditions: Results

- In total, found 167 of 188 conditions

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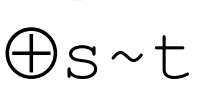


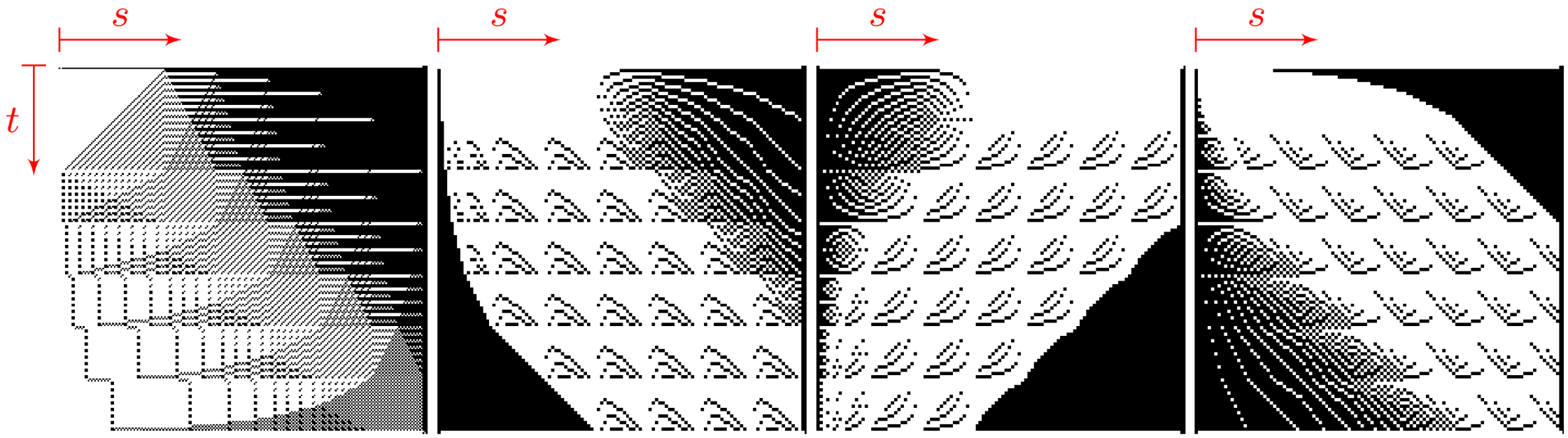
  - **14** choices for argument positions of $\oplus$
    - add_1, $sub_{1,2}$, $mult_1$, $div_{1,2}$, $rem_{1,2}$, $fma_{1,3}$, $sqrt_1$, abs_1, neg_1, $roundToIntegral_1$
  - **13** choices for ~
    - =, $\neq$, $\geq$, >, <, $\leq$, isNormal, isSubnormal, isZero, isInfinite, isNaN, isPositive, isNegative
  - **7** additional ICs for equations with only a relation, e.g. x>t

$\Rightarrow$ **14*13 + 7 = 188**

  - When applicable, ICs hold for each 5 rounding modes {RNE, RNA, RTP, RTN, RTZ} of operator

Finding Invertibility Conditions: Results

- In total, found 167 of 188 conditions



(a) $x + s \approx t$ (b) $x \cdot s \approx t$ (c) $s \div x \approx t$ (d) $x \div s \approx t$

$$t \approx (t \overset{\text{RTP}}{-} s) \overset{R}{+} s \vee t \approx (t \overset{\text{RTN}}{-} s) \overset{R}{+} s \vee s \approx t$$

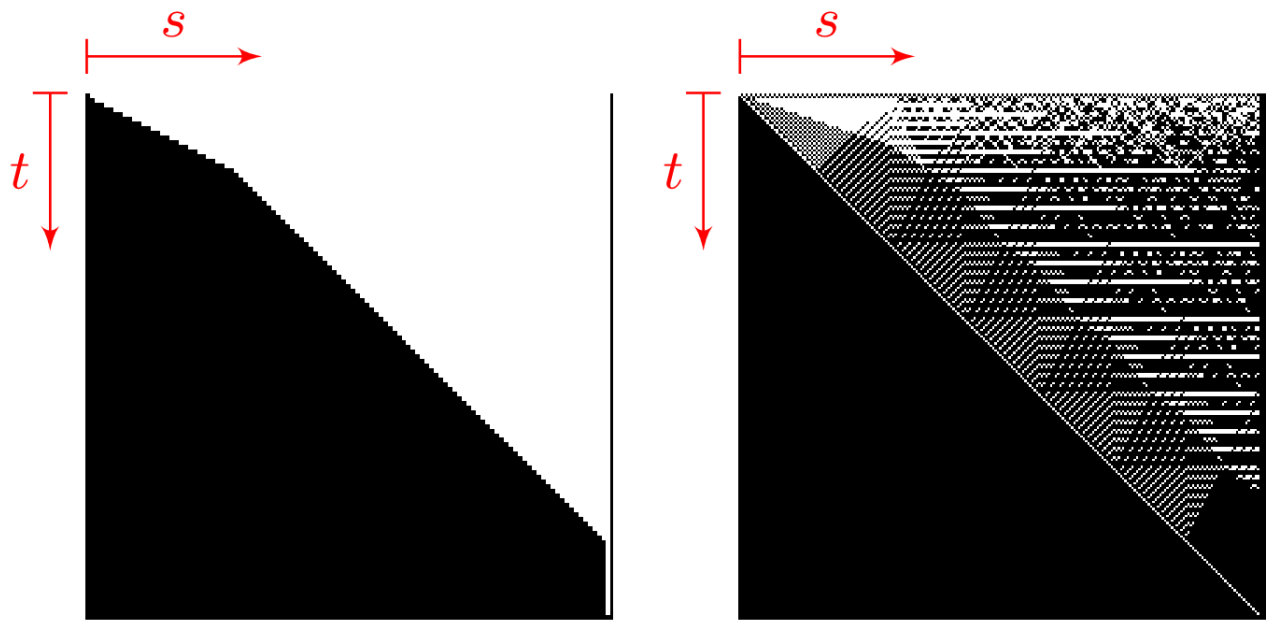
$$t \approx (t \overset{\text{RTP}}{\div} s) \overset{R}{\cdot} s \vee t \approx (t \overset{\text{RTN}}{\div} s) \overset{R}{\cdot} s \vee (s \approx \pm\infty \wedge t \approx \pm\infty) \vee (s \approx \pm 0 \wedge t \approx \pm 0)$$

$$t \approx (s \overset{\text{RTP}}{\cdot} t) \overset{R}{\div} s \vee t \approx (s \overset{\text{RTN}}{\cdot} t) \overset{R}{\div} s \vee (s \approx \pm\infty \wedge t \approx \pm 0) \vee (t \approx \pm\infty \wedge s \approx \pm 0)$$

$$t \approx s \overset{R}{\div} (s \overset{\text{RTP}}{\div} t) \vee t \approx s \overset{R}{\div} (s \overset{\text{RTN}}{\div} t) \vee (s \approx \pm\infty \wedge t \approx \pm\infty) \vee (s \approx \pm 0 \wedge t \approx \pm 0)$$

Finding Invertibility Conditions: Results

- Others were hard to find



(a) x rem $s \approx t$ (b) s rem $x \approx t$

$$|t \overset{\text{RTP}}{+} t| \leq |s| \vee |t \overset{\text{RTN}}{+} t| \leq |s| \vee \text{ite}(t \approx \pm 0, s \not\approx \pm 0, t \not\approx \pm\infty)$$

???

…behavior is pseudo-random!

Do Invertibility Conditions hold for all FP Formats?

- We **checked** all 167 invertibility conditions on other FP Formats
  - Targeted formats $\mathtt{FP}_{3,5}$, $\mathtt{FP}_{4,5}$, $\mathtt{FP}_{4,6}$, $\mathtt{FP}_{5,11}$, $\mathtt{FP}_{8,24}$, $\mathtt{FP}_{11,53}$
  - All invertibility conditions (appear to) generalize to all formats
    - **Successful in 94.5%** of the above cases using CVC4, Z3 (failures due to timeouts)
    - Approximately 32 days of compute time

- Future work: Format-Independent Verification!

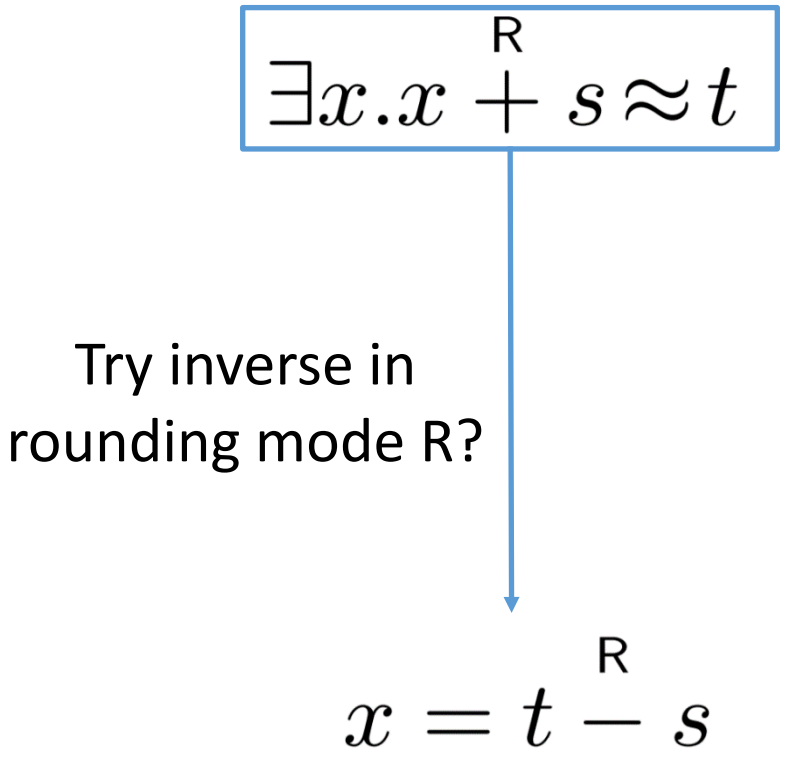
What can we do with Invertibility Conditions?

- Summary of intuition:

$$\exists x.x \overset{\mathsf{R}}{+} s \approx t$$

What can we do with Invertibility Conditions?

- Summary of intuition:



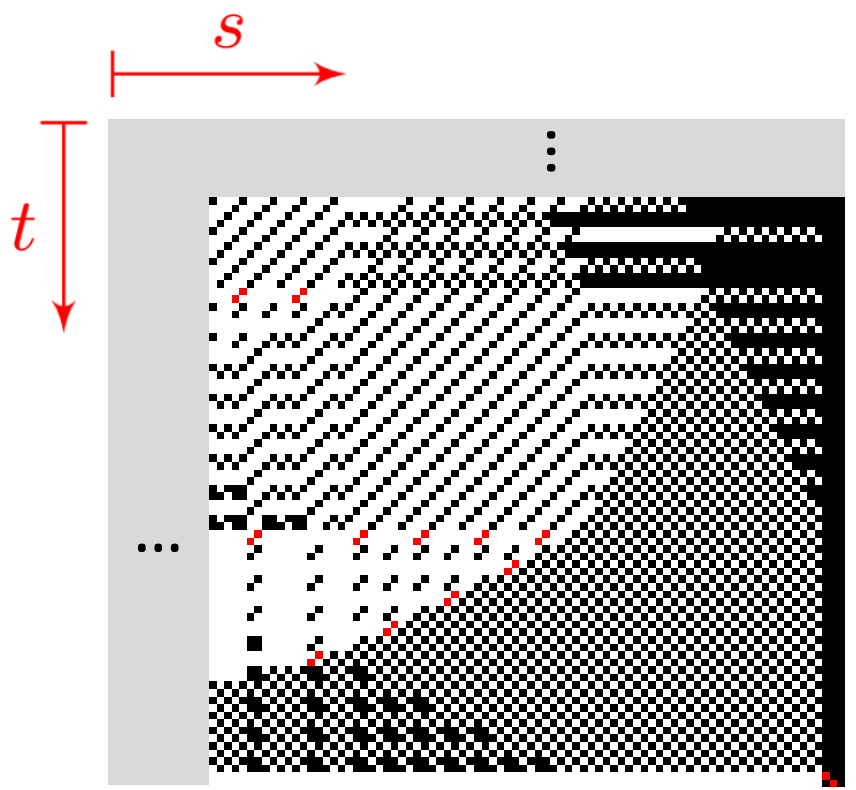
What can we do with Invertibility Conditions?

- Summary of intuition:

$$\exists x.x \overset{R}{+} s \approx t$$

Try inverse in rounding mode R?

$$x = t \overset{R}{-} s$$



red = wrong (showing R=RNE)

What can we do with Invertibility Conditions?

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What can we do with Invertibility Conditions?

- Summary of intuition:

$$\boxed{\exists x.x \overset{\text{R}}{+} s \approx t}$$

$$\Updownarrow$$

$$(t - s) \overset{\text{RTP}}{\overset{R}{+}} s \approx t \vee (t - s) \overset{\text{RTN}}{\overset{R}{+}} s \approx t \vee s \approx t$$

What can we do with Invertibility Conditions?

- Summary of intuition:

$$\overset{\text{R}}{\exists x.x + s \approx t}$$

$$\Updownarrow$$

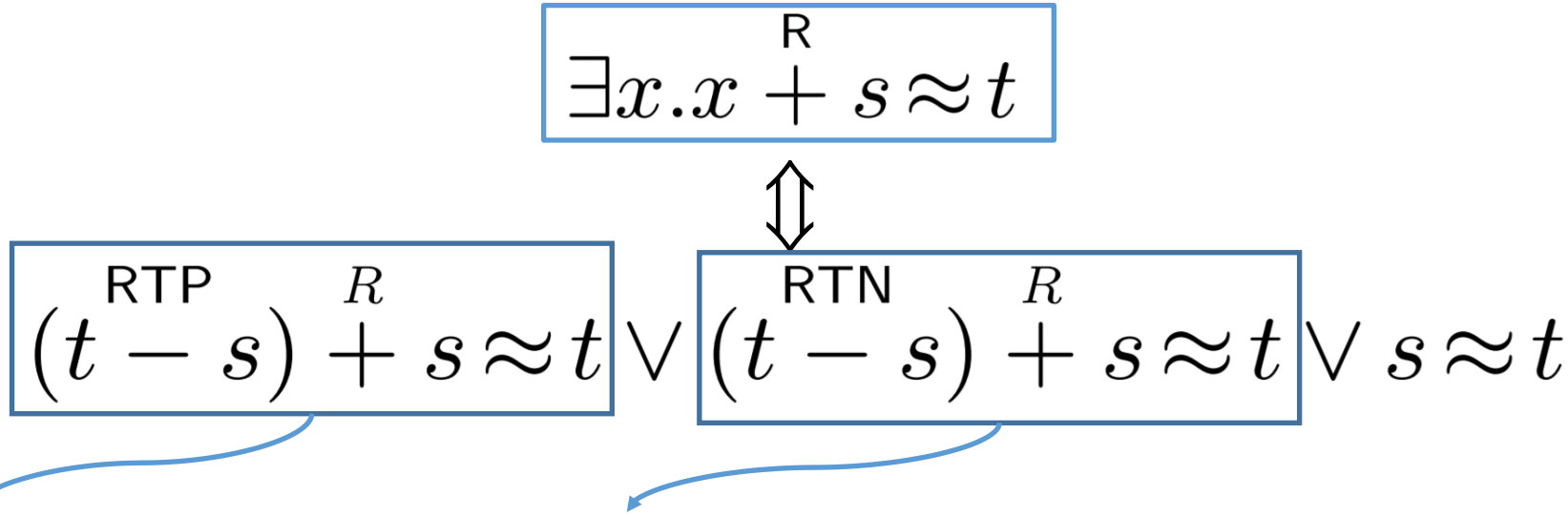
$$\overset{\text{RTP}}{(t - s)} \overset{R}{+} s \approx t \vee \overset{\text{RTN}}{(t - s)} \overset{R}{+} s \approx t \vee s \approx t$$

Try rounding towards positive

$$x \overset{\text{RTP}}{=} t - s$$

What can we do with Invertibility Conditions?

- Summary of intuition:



$$x = \overset{RTP}{t - s} \qquad x = \overset{RTN}{t - s}$$

What can we do with Invertibility Conditions?

- Summary of intuition:

$$\exists x.x \overset{\mathrm{R}}{+} s \approx t$$

$$\Updownarrow$$

$$(t - s) \overset{\mathrm{RTP}}{\overset{R}{+}} s \approx t \vee (t - s) \overset{\mathrm{RTN}}{\overset{R}{+}} s \approx t \vee s \approx t$$

Try rounding towards positive

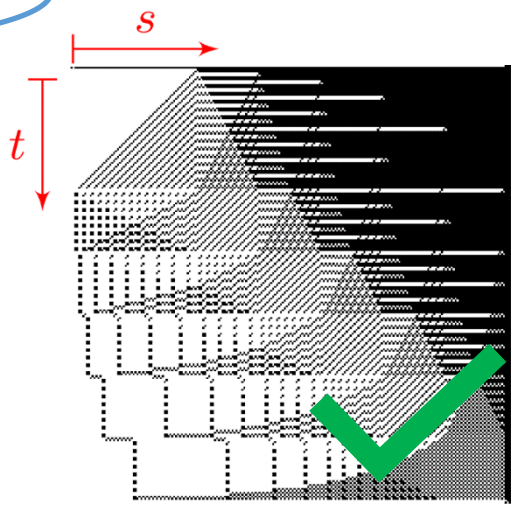
Try rounding towards negative

Corner case, try zero

$$x = t \overset{\mathrm{RTP}}{-} s$$

$$x = t \overset{\mathrm{RTN}}{-} s$$

$$x = \pm 0$$



What can we do with Invertibility Conditions?

- Similar intuition for many other invertibility conditions:

Equation	Set of Possible Solutions
$x \overset{\mathsf{R}}{\cdot} s \approx t$	$x \in \{t \overset{\mathsf{RTP}}{\div} s, t \overset{\mathsf{RTN}}{\div} s\}$
$x \overset{\mathsf{R}}{\div} s \approx t$	$x \in \{t \overset{\mathsf{RTP}}{\cdot} s, t \overset{\mathsf{RTN}}{\cdot} s\}$
$s \overset{\mathsf{R}}{\div} x \approx t$	$x \in \{s \overset{\mathsf{RTP}}{\div} t, s \overset{\mathsf{RTN}}{\div} t\}$
$\overset{\mathsf{R}}{\sqrt{x}} \approx t$	$x \in \{t \overset{\mathsf{RTP}}{\cdot} t, t \overset{\mathsf{RTN}}{\cdot} t\}$
$\mathsf{fma}(x, s, t) \approx \pm 0$	$x \in \{-(t \overset{\mathsf{RTP}}{\div} s), -(t \overset{\mathsf{RTN}}{\div} s)\}$

+ corner cases for ± 0, $\pm\infty$, NaN

Ongoing Work

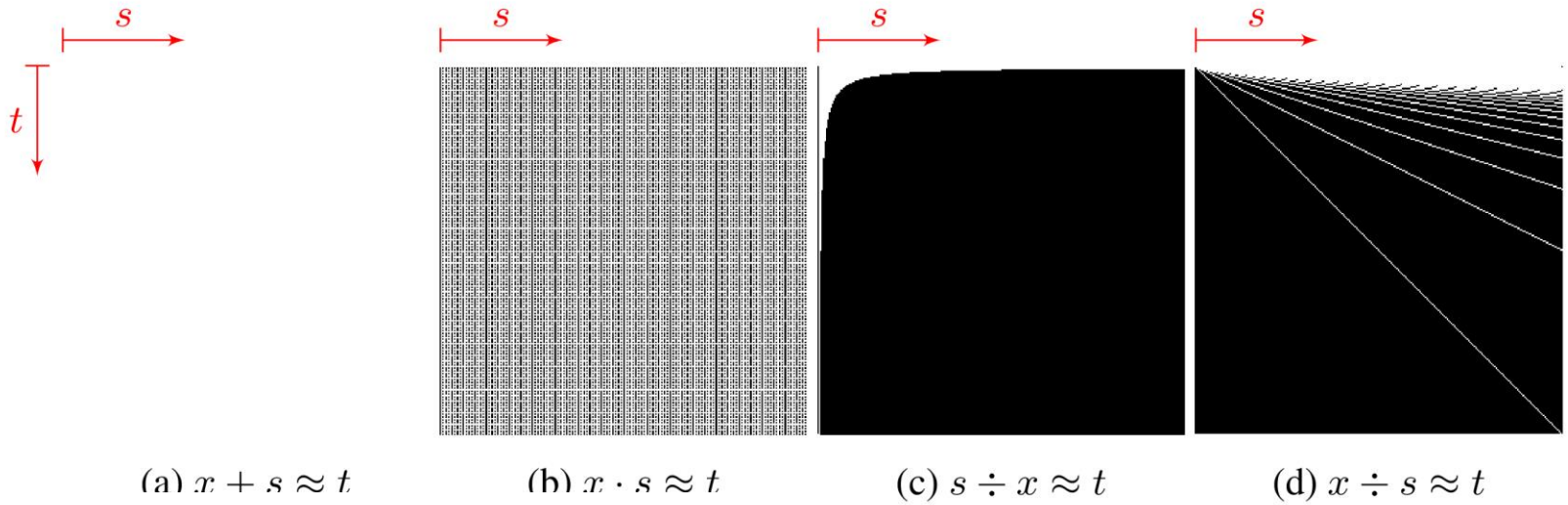
- Further rewrite rules, test case generation
- Solving algorithms based on invertibility conditions
  - Quantifier Elimination
  - Local search techniques **[Niemetz et al CAV2016]**
  - Quantifier Instantiation for Synthesis (for BV, FP, …)
    - Quantifier instantiation procedures ⇔ synthesis procedures
- Bit-width independent verification of invertibility conditions
  - Initial work “Toward Bit-Width Independent Proofs in SMT Solvers” **[CADE 2019]**
- Other applications: Syntax-guided optimization of FP code?
- …Your application here

Thanks for Listening!

- Techniques in talk available in SMT solver cvc5
  - Open-source : https://cvc5.github.io/

- Questions?

Visualizing Invertibility Conditions for Bit-Vectors



(a) $x + s \approx t$ (b) $x \cdot s \approx t$ (c) $s \div x \approx t$ (d) $x \div s \approx t$

(a) x rem $s \approx t$ (b) s rem $x \approx t$

White = IC is true

Black = IC is false

(shown for 8-bit)