Outline

Motivating Example

Representation Theory of Finite Groups

Brauer Tree Algebras

Representation Theory of Special Linear Groups
Motivating Example

$p$ prime, $G$ cyclic group order $p^n$, $k$ algebraically closed field, $\text{char}(k) = p$
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Describe ALL the indecomposable representations
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Describe ALL the indecomposable representations

Indecomposable representations given by Jordan blocks

\[
\begin{pmatrix}
1 & 1 & 0 & \cdots & 0 & 0 \\
0 & 1 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \cdots & 1 & 0 \\
0 & 0 & \cdots & 1 & 1 \\
0 & 0 & \cdots & 0 & 1 \\
\end{pmatrix}\ 
\]

for $1 \leq j \leq p^n$ (see *Local Representation Theory*, J.L. Alperin)
Motivating Example

Indecomposable representations

$kG$ is a Brauer tree algebra for the Brauer tree $\circ \bullet$ with multiplicity $m = p^n - 1$.
Motivating Example

Indecomposable representations

Uniserial $kG$-modules of length $j$ for $1 \leq j \leq p^n$, with trivial composition factors
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\[ \circ \quad \quad \bullet \]

with multiplicity $m = p^n - 1$
Definition

Let $G$ be a finite group and let $k$ be a field. The group ring is defined to be the set

$$kG = \left\{ \sum_{g \in G} a_g g \mid a_g \in k \right\}$$

with multiplication given by group multiplication. This space is a vector space of dimension $|G|$ over $k$. 
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Definition

A representation of a finite group $G$ over a field $k$ is a $kG$-module.
Modular Representation Theory

Representations of a group over a field of prime characteristic
Modular Representation Theory

Representations of a group over a field of prime characteristic
Study of $kG$-modules
Modular Representation Theory

Representations of a group over a field of prime characteristic
Study of $kG$-modules

Theorem (Drozd, Crawley-Boevey)

A finite dimensional algebra $\Lambda$ over an algebraically closed field is one of the following mutually exclusive types:

1. Finite (finitely many indecomposable modules)
2. Tame (infinitely many indecomposable modules, can be parametrized)
3. Wild (A full subcategory of $\Lambda$-mod is equivalent to $k\langle x, y \rangle$-mod)

Theorem (Higman)
Let $G$ be a finite group and let $k$ be an algebraically closed field of characteristic $p$. Then, $kG$ is of finite representation type if and only if $G$ has cyclic Sylow $p$-subgroups.
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Blocks and Brauer Trees

$$kG = B_1 \oplus \cdots \oplus B_m$$

Unique decomposition into indecomposable subalgebras
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Each block $B$ has a defect group $D \leq G$, measures deviation of $B$ from being semisimple as an algebra
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Unique decomposition into indecomposable subalgebras

Each block $B$ has a defect group $D \leq G$, measures deviation of $B$ from being semisimple as an algebra

Theorem (See Chapter V, Alperin)

If $B$ is a block of $kG$ with cyclic defect group, then $B$ is a Brauer tree algebra
Let $M$ be a $kG$-module.
Module Definitions

Let $M$ be a $kG$-module.

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\[
0 \subset \text{soc}(M) \subset \text{soc}^2(M) \subset \cdots \subset \text{soc}^{m-1}(M) \subset \text{soc}^m(M) = M \n\]
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Definition

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Note that

\[ \text{simple} \implies \text{indecomposable} \]

but

\[ \text{indecomposable} \nLeftrightarrow \text{simple} \]
Theorem (Maschke)

Let $G$ be a finite group and let $k$ be a field of characteristic $p$. The group algebra $kG$ is semisimple if and only if $p 
mid |G|$. 
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Representation Theory of Finite Groups

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- If $p 
mid |G|$ (or $\text{char}(k) = 0$), study the simple representations
- If $p 
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Brauer Trees

Definition

A *Brauer tree* is a finite unoriented connected graph $T = (T_0, T_1)$ with no loops or cycles satisfying the additional properties:

1. There is an exceptional vertex with a multiplicity $m \geq 1$
2. For each vertex $v$, there is a cyclic ordering of edges incident with $v$
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Notation and conventions:

- $T_0$ is the vertex set
- $T_1$ is the edge set
- The exceptional vertex will be solid or bold; the other vertices will not be filled in or plain text
- We view the graph in the plane and assume a counterclockwise orientation of the edges
- Notation for a Brauer tree: $T = (T_0, T_1, m)$
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Example

\[ T = (T_0, T_1, m) \]

Vertex 4 is the exceptional vertex and has multiplicity 2.
$T = (T_0, T_1, m)$

$T_0 = \{1, 2, 3, 4, 5\}$
Example

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\[ T_1 = \{a, b, c, d\} \]
\[ m = 2 \]
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Example

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Orientation at 2

\[ b < a \text{ and } a < b \]
Example

\[ T = (T_0, T_1, m) \]

Orientation at 3

\[ c < b < d < c \]
Example

\[ T = (T_0, T_1, m) \]

Orientation at 4

\[ c < c \]
Let $T = (T_0, T_1, m)$ be a Brauer tree.
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There is an arrow $a : i \rightarrow j$ if $i < j$ and $j$ is the “next” edge after $i$. In this case, $a$ is said to be given by the successor relation $(i, j)$. 
Example

Recall \( T = (T_0, T_1, m) \)

\[ Q = (Q_0, Q_1) \]
Let $v \in T_0$ be a vertex.

- If $v$ is not exceptional and $\#(\text{edges adjacent to } v) \geq 2$, then there is an oriented cycle in $Q$, unique up to cyclic permutation.

- If $v$ is exceptional and $\#(\text{edges adjacent to } v) \cdot m \geq 2$, then there is an oriented cycle in $Q$, unique up to cyclic permutation.

- Call this cycle the \textit{special cycle} at $v$.

- If the cycle starts at $i \in Q_0 = G_1$, call it the \textit{special $i$-cycle} at $v$. 
Special Cycles

Let \( v \in T_0 \) be a vertex.

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\[ T = (T_0, T_1, m) \]

\[ Q = (Q_0, Q_1) \]

Special cycle at 2
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Special cycle at 3
Example

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Special cycle at 4
Let $T = (T_0, T_1, m)$. There are two ways of building an algebra over a field $k$ associated to $T$. 

1. Get the associated quiver $Q$, define certain relations $I$, and define $\Gamma_T = kQ/I$ to be the path algebra with relations.

2. Define an algebra $\Lambda_T$ over $k$ by defining the projective indecomposable $\Lambda$-modules via the graph $T$. 

These two methods give the same result. That is, $\Gamma_T \cong \Lambda_T$. 
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$\Gamma_T$, Path Algebra with Relations

Special $a$-cycle at 2: $\beta \alpha$
Special $b$-cycle at 2: $\alpha \beta$
Special $c$-cycle at 3: $\epsilon \delta \gamma$
Special $b$-cycle at 3: $\gamma \epsilon \delta$
Special $d$-cycle at 3: $\delta \gamma \epsilon$
Special $c$ cycle at 4: $\iota$
\[ \Gamma_T, \text{ Path Algebra with Relations} \]

Relations
\[ \alpha \beta = \gamma \epsilon \delta \]
\[ \epsilon \delta \gamma = \iota^2 \]
\[ \alpha \beta \alpha = \beta \alpha \beta = \gamma \epsilon \delta \gamma = \delta \gamma \epsilon \delta = \epsilon \delta \gamma \epsilon = \iota^3 = 0 \]
\[ \delta \alpha = \beta \gamma = \iota \epsilon = \gamma \iota = 0 \]
$kQ/I$ is a $k$-vector space with allowable paths given by

- $\alpha, \beta, \gamma, \delta, \epsilon, \iota$
- $\beta\alpha, \alpha\beta, \epsilon\delta, \gamma\epsilon, \delta\gamma, \iota^2$
- $\delta\gamma\epsilon$

Multiply by concatenating paths
$T = (T_0, T_1, m)$

For each edge $i$, get a module $M_i$ as follows

$$M_a = \begin{pmatrix} a \\ b \\ a \end{pmatrix}, \quad M_b = \begin{pmatrix} a & b & d \\ c \\ b \end{pmatrix}, \quad M_c = \begin{pmatrix} b & c \\ d & c \end{pmatrix}, \quad M_d = \begin{pmatrix} c & d \\ b & d \end{pmatrix}$$

Define $\Lambda_T$ so that the projective indecomposable $\Lambda$-modules are $M_a$, $M_b$, $M_c$, and $M_d$. 
Overview

$G$ finite group

\[ kG = B_1 \oplus \cdots \oplus B_m \]

Goal: Understand indecomposable modules for a block $B$
Overview

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Suppose $B$ has cyclic defect group
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Suppose $B$ has cyclic defect group

\[ B \]

\[ \downarrow \]

Brauer Tree

\[ \downarrow \]

Quiver with Relations
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Goal: Understand indecomposable modules for a block $B$ 
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\[ B \]

\[ \Downarrow \]

Brauer Tree

\[ \Downarrow \]

Quiver with Relations

\[ \Downarrow \]

Butler and Ringel (1987)

\[ \Downarrow \]

String Modules
Let $\mathbb{F}_p$ denote the finite field with $p$ elements.

Define

$$SL_2(\mathbb{F}_p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{Mat}_{2\times2}(\mathbb{F}_p) \mid a, b, c, d \in \mathbb{F}_p, ad - bc = 1 \right\}.$$ 

$$|SL_2(\mathbb{F}_p)| = \frac{1}{2} p(p - 1)(p + 1)$$

Goal: Understand the indecomposable representations of $SL_2(\mathbb{F}_p)$ over a field of characteristic $p$. 
$k[SL_2(\mathbb{F}_p)] = B_0 \oplus B_{\text{odd}} \oplus B_{\text{even}}$

There are $p$ simple modules, one for each dimension $1, \ldots, p$

$B_0$ trivial defect group

$B_{\text{odd}}$ cyclic defect group, contains odd dimensional simple modules

$B_{\text{even}}$ cyclic defect group, contains even dimensional simple modules
Assuming $p \equiv -1 \mod 4$

$B_{\text{odd}}$

1 $\cdots$ $p-2$ 3 $\cdots$ $(p-1)/2$

Multiplicity 2
Assuming $p \equiv -1 \mod 4$

$B_{\text{odd}}$

\[
\begin{array}{cccccccc}
1 & \bullet & p-2 & \bullet & 3 & \bullet & \ldots & \bullet & (p-1)/2 \\
\end{array}
\]

Multiplicity 2

\[
\begin{array}{cccccccc}
\alpha_1 & \alpha_2 & \alpha_3 & \ldots & \alpha (p-3)/2 & \alpha \\
\beta_1 & \beta_2 & \beta_3 & \beta (p-3)/2 & &
\end{array}
\]
$B_{\text{odd}} \text{ for } p = 7$

Relations

$\alpha_1 \beta_1 \alpha_1, \beta_1 \alpha_1 \beta_1, \alpha_2 \beta_2 \alpha_2, \beta_2 \alpha_2 \beta_2$

$\alpha^3$

$\alpha_1 \beta_1 - \beta_2 \alpha_2, \alpha_2 \beta_2 - \alpha^2$

$\alpha_2 \alpha_1, \beta_1 \beta_2, \alpha \alpha_2, \beta_2 \alpha$
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$\alpha_2 \alpha_1, \beta_1 \beta_2, \alpha \alpha_2, \beta_2 \alpha$
$B_{\text{odd}}$ for $p = 7$

Only allowable loops in $kQ/I$ are $\beta_1\alpha_1$ and $\alpha$

For rest of talk, let $\Lambda = kQ/I$
Strings

$\mathcal{Q}$ quiver, $\mathcal{I}$ admissible ideal
Strings

$Q$ quiver, $I$ admissible ideal

$\alpha$ arrow, define $\alpha^{-1}$ formal inverse, “flipped” arrow
Strings

Q quiver, \( I \) admissible ideal
\( \alpha \) arrow, define \( \alpha^{-1} \) formal inverse, “flipped” arrow

**Definition**

A *string* of length \( m \) is a finite concatenation of arrows and inverses of arrows

\[ c_1 c_2 \cdots c_{m-1} c_m \]

so that

- \( c_{i+1} \neq c_i^{-1} \) for all \( 1 \leq i \leq m \)
- No subpath \( c_i c_{i+1} \cdots c_{i+t} \) or its inverse belongs to \( I \)
Strings

$Q$ quiver, $I$ admissible ideal

$\alpha$ arrow, define $\alpha^{-1}$ formal inverse, “flipped” arrow

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Visualize

$\alpha_1 \alpha_2 \alpha_3^{-1} \alpha_4 \alpha_5^{-1} \alpha_6^{-1} \alpha_7$

as

\[\bullet \leftarrow \alpha_1 \bullet \leftarrow \alpha_2 \bullet \leftarrow \alpha_3 \rightarrow \bullet \leftarrow \alpha_4 \bullet \rightarrow \bullet \leftarrow \alpha_5 \bullet \rightarrow \bullet \leftarrow \alpha_6 \bullet \rightarrow \bullet \leftarrow \alpha_7 \bullet\]
Strings in $\Lambda$

Butler and Ringel, “Auslander-Reiten Sequences with Few Middle Terms and Applications to String Algebras,” Communications in Algebra (1987)
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\{\text{Indecomposable } \Lambda\text{-modules}\} \leftrightarrow \{\text{Strings in } \Lambda\}
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\[
\{\text{Indecomposable } \Lambda\text{-modules}\} \leftrightarrow \{\text{Strings in } \Lambda\}
\]

\[
1 \xleftarrow{\beta_1} 5 \xrightarrow{\alpha_1} 3 \xleftarrow{\beta_2} \alpha
\]
Strings in Λ

Butler and Ringel, “Auslander-Reiten Sequences with Few Middle Terms and Applications to String Algebras,” Communications in Algebra (1987)

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\{\text{Indecomposable } \Lambda\text{-modules}\} \longleftrightarrow \{\text{Strings in } \Lambda\}\]

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\]

<table>
<thead>
<tr>
<th>Length 0</th>
<th>Length 1</th>
<th>Length 2</th>
<th>Length 3</th>
<th>Length 4 and 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>e₁</td>
<td>α₁</td>
<td>β₁α₁</td>
<td>αβ⁻¹₁₁</td>
<td>α²⁻¹_αβ⁻¹_α₁</td>
</tr>
<tr>
<td></td>
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<tr>
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<td>β₁α⁻¹₂</td>
<td>β₁α⁻¹²αβ⁻¹_α₁</td>
</tr>
</tbody>
</table>
Auslander-Reiten Theory

Know indecomposable modules

Goal: Understand maps between indecomposable modules

Subgoal: Understand “irreducible” maps between indecomposable modules

Definition

Let $f : A \rightarrow B$ be a module homomorphism. Then, $f$ is irreducible if $f$ is not an isomorphism and if $f = hg$ is a factorization of $f$, either $g$ is a split monomorphism or $h$ is a split epimorphism.
Auslander-Reiten Theory

The Auslander-Reiten quiver of $\Lambda$ is a quiver defined by

Vertices: indecomposable $\Lambda$-modules

Arrows: irreducible maps between indecomposable modules

Can be built from “almost split exact sequences”
Butler and Ringel (1987) give method for getting almost split exact sequences

**Definition**

Let $C$ be a string.

1. A string $C$ *starts on a peak* if there does not exist an arrow $\beta$ so that $C\beta$ is a string.
2. A string $C$ *starts in a deep* if there does not exist an arrow $\gamma$ so that $C\gamma^{-1}$ is a string.
3. A string $C$ *ends on a peak* if there does not exist an arrow $\beta$ so that $\beta^{-1}C$ is a string.
4. A string $C$ *ends in a deep* if there does not exist an arrow $\gamma$ so that $\gamma C$ is a string.

Build almost split exact sequences from strings
Example

String $\alpha \beta_2^{-1}$ does not end on a peak, does not start on a peak

$\beta_1 \alpha_2^{-1} \alpha \beta_2^{-1}$ “maximal” string to the left

$\alpha \beta_2^{-1} \alpha_1$ “maximal” string to the right

$\beta_1 \alpha_2^{-1} \alpha \beta_2^{-1} \alpha_1$ “maximal” string in both directions

Get almost split exact sequence

$$0 \to \alpha \beta_2^{-1} \to \beta_1 \alpha_2^{-1} \alpha \beta_2^{-1} \oplus \alpha \beta_2^{-1} \alpha_1 \to \beta_1 \alpha_2^{-1} \alpha \beta_2^{-1} \alpha_1 \to 0$$
Auslander-Reiten Quiver
Connection to Current Research

$X$ smooth projective curve over algebraically closed field $k$ of characteristic $p$, $G$ finite group acting on $X$

For $m > 1$, define $H^0(X, \Omega_X^m)$, space of holomorphic polydifferentials (Representation of $G$)

Special Case:

- $\ell \neq p$ prime, $X(\ell)$ modular curve of level $\ell$
- $X_p(\ell)$ reduction of $X(\ell)$ modulo $p$
- $G = PSL(2, \mathbb{F}_\ell)$ acts on $X_p(\ell)$
- Understand $H^0(X_p(\ell), \Omega_{X_p(\ell)}^m)$ as a representation of $PSL(2, \mathbb{F}_\ell)$
Blocks of $\text{PSL}(2, \mathbb{F}_\ell)$ over $k$ look like

where the exceptional vertex has multiplicity $\frac{p^n-1}{2}$


Frauke M. Bleher, Ted Chinburg, and Artistides Kontogeorgis. “Galois structure of the holomorphic differentials of curves”. In progress. 2017.