Projective Modules, Indecomposable Modules, and Simple Modules*

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Let $G$ be a finite group and let $k$ be a field. The goal of this document is to prove that the projective indecomposable $kG$-modules form a basis for Grothendieck group of finitely generated projective $kG$-modules. Moreover, the projective indecomposable modules are in bijective correspondence with the simple modules. We strive to present a self contained discussion of projective indecomposable modules and the bijection between projective indecomposable modules and simple modules. For references see [2] Chapter X, Section 7, [1] Chapter 3, Section 12, [5], and [3].

Preliminary Definitions

Definition 1. Let $R$ be a ring. The radical of $R$, denoted $\text{rad}(R)$, is defined to be the collection of all elements of $R$ that annihilate every simple $R$-module. An equivalent definition is that $\text{rad}(R)$ is the intersection of all maximal left ideals of $R$.

Definition 2. Let $M$ be an $R$-module. The $R$-module $\text{rad}(R)M$ is defined to be the radical of $M$ and is denoted by $\text{rad}(M)$.

Definition 3. An $R$-module $M$ is semisimple if $M$ is a direct sum of simple modules, or equivalently, if every submodule of $M$ is a direct summand.

Definition 4. An algebra $R$ is called local if every element is either invertible or nilpotent.

Semisimplicity

We first prove a few results classifying semisimple modules.

Lemma 1. A module is $M$ is semisimple if and only if $\text{rad}(M) = 0$.

Proof. If $\text{rad}(M) = 0$, then $\text{rad}(R)M = 0$ and $M$ becomes an $R/\text{rad}(R)$ module. Since $R/\text{rad}(R)$ is a semisimple ring, $M$ is a semisimple module. If $M$ is semisimple, then $M$ is a direct sum of simple modules. Since simple modules are annihilated by $\text{rad}(R)$ by definition, $\text{rad}(R)S_i = 0$ for all simple modules and therefore, we have that $\text{rad}(R)M = 0$. By definition of the radical of $M$, we have that $\text{rad}(M) = 0$. \hfill $\Box$

Lemma 2. A module $M$ is semisimple if and only if the intersection of all maximal submodules of $M$ is $0$.

Proof. Let $M$ be semisimple and write

$$M = \bigoplus_{i=1}^{n} S_i$$

for simple modules $S_i$. Define

$$N_j = \bigoplus_{i \neq j} S_i$$

* A note that was started as an investigation of Chapter 14, Section 2 of [4]. I used various resources and wrote this note to attempt to fully understand the projective modules and the indicated correspondence. There is no original material. I just collected and reorganized material to make the ideas make sense to me.
and note that, since $M/N_j \cong S_j$, $N_j$ is a maximal submodule of $M$. The intersection of the $N_j$ is zero since $S_j \cap \left( \bigoplus_{i \neq j} S_i \right) = 0$ by definition of the direct sum. Conversely, suppose that the intersection of all maximal submodules of $M$ is 0. Then, $\bigcap_{i=1}^m M_i = 0$, where $M_i$ are the maximal submodules of $M$. We have a well defined isomorphism

$$M \rightarrow \bigoplus_{i=1}^m M/M_i$$

given by $x \mapsto (x + M_1, \ldots, x + M_m)$. This map has kernel equal to the intersection of the $M_i$, which is equal to zero. Therefore, $M$ is a submodule of a direct sum of simple modules and is therefore semisimple. □

Applying Lemma 2 above to a quotient, we get the following corollary.

**Corollary 1.** A module $M/N$ is semisimple if and only if the intersection of all maximal submodules of $M$ is contained in $N$.

**Proof.** Note that the maximal submodules of $M/N$ are in bijective correspondence with the maximal submodules of $M$ containing $N$. Applying the Lemma 2, we have that $M/N$ is semisimple if and only if the intersection of all maximal submodules of $M/N$ is 0. If the intersection of all maximal submodules of $M/N$ is 0, then the intersection of the preimages of these submodules under the natural projection map in contained in $N$. Therefore, the intersection of all maximal submodules of $M$ is still contained in $N$. Conversely, if the intersection of all maximal submodules of $M$ is contained in $N$, then the intersection of all maximal submodules of $M$ containing $N$ is also contained in $N$ and we have that the intersection of all maximal submodules of $M/N$ is 0. □

**Lemma 3.** The radical of $M$ is equal to the intersection of all maximal submodules of $M$.

**Proof.** Let $\text{rad}(M)$ denote the radical of $M$ and let $I$ denote the intersection of all maximal submodules of $M$. By the Corollary 1, $M/N$ is semisimple if and only if $I \subseteq N$. By Lemma $\text{rad}$, $M/N$ is semisimple if and only if $\text{rad}(R)(M/N) = 0$, that is, $\text{rad}(R)M \subseteq N$. We know that $M/I$ is semisimple by the Corollary 1. Since $M/I$ is semisimple, $\text{rad}(M) \subseteq I$. We next want to show that $\text{rad}(\text{rad}(M)) = 0$, which will show that $M/\text{rad}(M)$ is semisimple. Consider an element in $x \in \text{rad}(\text{rad}(M))$. Then, $x = r(m + \text{rad}(M))$, where $r,s$ for all $s \in S$ for all simple modules $S$ and $m \in M$. By definition of $\text{rad}(M)$, $rm \in \text{rad}(M)$, which shows that $x = 0$. Thus, $M/\text{rad}(M)$ is semisimple. Therefore, $I \subseteq \text{rad}(M)$. We have shown that the radical of $M$ is equal to the intersection of all maximal submodules of $M$. □

**Proposition 1.** The submodule $\text{rad}(P)$ of $P$ is the smallest submodule of $P$ with semisimple quotient.

**Proof.** First note that $P/\text{rad}(P)$ is semisimple since $\text{rad}(P)$ is the intersection of all maximal submodules of $P$ so that we have semisimplicity by the above corollary. Also, if $N$ is a submodule of $P$ so that $P/N$ is semisimple, then $\text{rad}(P) \subseteq N$.

Note that the previous proposition implies that $P/\text{rad}(P)$ is the largest semisimple quotient of $P$.

**Projective Cover**

Given a module $M$ over an Artinian ring, there always exists a projective cover of $M$. Given a projective module $P$, is it the projective cover of some module and, if so, can we characterize that module?

**Proposition 2.** Let $P$ be a projective module. Then, $P$ is a projective cover for $P/\text{rad}(P)$.

**Proof.** Let $\pi : P \rightarrow \text{Prad}(P)$ be the natural projection homomorphism. Since $\pi$ is surjective, we must just show that $\pi$ is essential. Assume, the sake of contradiction, that $Q \subseteq P$ is a proper submodule satisfying $\pi(Q) = Q/\text{rad}(P) = P/\text{rad}(P)$. If $P/N$ is simple for some submodule $N$, then it is semisimple and therefore is contained in $P/\text{rad}(P)$. We then get a well defined surjective map $P/\text{rad}(P) \rightarrow P/N$ for all simple quotients of $P$. By assumption, $\pi(Q) = P/\text{rad}(P)$, so there is also a well defined surjective map $Q \rightarrow P/N$. □
Now, $P/N$ is simple if and only if $N$ is a maximal submodule of $P$. We have shown that $Q$ surjects onto any simple quotient of $P$. Then, this statement is equivalent to $Q$ surjecting onto a quotient of $P$ by any maximal submodule. So, $Q$ is not contained in any maximal submodule of $P$, which contradicts the fact that $Q$ is a proper submodule of $P$. Therefore, $\pi$ is essential. Thus, $P$ is a projective cover for $P/\text{rad}(P)$.

**Decomposition**

We next claim that any projective module can be decomposed into a direct sum of indecomposable projective modules. We will essentially apply the Krull-Schmidt Theorem and provide a proof for the decomposition as well as comment on the projectivity of the direct sum decomposition.

**Proposition 3.** Let $M$ be a finitely generated module over an Artinian ring. Then, $M$ can be written as a finite direct sum of indecomposable submodules.

**Proof.** Note that since $M$ is finitely generated module over an Artinian ring, $M$ is both a Noetherian module and an Artinian module. Assume, for the sake of contradiction, that $M$ cannot be decomposed into a direct sum of indecomposable submodules. Then, write $M = N_0 \oplus N'_0$, where $N'_0$ cannot be decomposed into a direct sum of indecomposable submodules. Write $N'_0 = N_1 \oplus N'_1$, where $N'_1$ cannot be decomposed into a direct sum of indecomposable submodules. Continuing this process inductively, we get an infinite decreasing chain of submodules of $M$,

$$N'_0 \supseteq N'_1 \supseteq N'_2 \supseteq \cdots \supseteq N'_{k-1} \supseteq N'_k \supseteq N'_{k+1} \supseteq \cdots$$

contradicting the assumption that $M$ is Artinian. Thus, $M$ can be decomposed into a direct sum of indecomposable submodules.

**Proposition 4.** Let $P$ be a finitely generated projective module over an Artinian ring. Then, $P$ can be written as a finite direct sum of indecomposable projective submodules.

**Proof.** Apply the previous proposition to write $P$ as a direct sum of indecomposable submodules. Since $P$ is projective, any direct summand of $P$ is projective so that $P$ is actually a direct sum of indecomposable projective submodules.

Moreover, this decomposition is unique up to isomorphism.

**Proposition 5.** Let

$$M = \bigoplus_{i=1}^n N_i = \bigoplus_{j=1}^m M_j$$

be two decompositions of the module $M$ into indecomposable modules. Then, $n = m$ and, up to permutation of the factors, $N_i \cong M_i$ for all $1 \leq i \leq n$.

**Proof.** We have projection maps $\pi_i : M \to N_i$ and $\psi_j : M \to M_j$. Consider $N_1$ and consider the composition of projection maps, $\pi_1 \psi_j$ and $\psi_j \pi_1$. Recall that $\sum_{j=1}^m \psi_j = \text{id}_M$. Therefore,

$$\sum_{j=1}^m \pi_1 \psi_j \psi_j \pi_1 = \text{id}_{N_1}.$$
indecomposable, either $e = 1$ or $e = 0$. Since $(\alpha_j\beta_j)^{-1}\alpha_j\beta_j = 1^2 = 1 \neq 0$, $e \neq 0$. Therefore, $e = id_{M_2}$. So, $\alpha_j$ is injective and $\beta_j$ is surjective. Since $\alpha_j\beta_j$ is invertible, $\beta_j$ is injective and $\alpha_j$ is surjective. Thus, $\alpha_j$ and $\beta_j$ are isomorphisms. After renumbering, we can assume that we have an isomorphism $N_1 \to M_1$ induced by $\pi_1$. Then,

$$\text{Ker}(\pi_1) = \bigoplus_{i=2}^{n} N_i$$

so that $M_1 \cap \bigoplus_{i=2}^{n} N_i = 0$. Also, $\pi_1(M_1) = N_1$. Therefore,

$$M \cong M_1 \oplus \bigoplus_{i=2}^{n} N_i = M_1 \oplus \bigoplus_{i=2}^{m} M_j.$$ 

Therefore,

$$\bigoplus_{i=2}^{n} N_i \cong \bigoplus_{i=2}^{m} M_j.$$ 

Continuing by induction, we get that $n = m$. Also, up to permutation of the factors, $N_i \cong M_i$ for all $1 \leq i \leq n$. 

\textbf{Correspondence}

We now show that there is a bijective correspondence between projective indecomposable modules and simple modules. In particular, the projective indecomposable modules are the projective covers of the simple modules.

We first prove the classic Fitting's Lemma for finitely generated modules over Artinian rings. The result holds for any finite length module over an arbitrary ring.

\textbf{Lemma 4.} Let $M$ be a finitely generated module over an Artinian ring $R$ and let $\phi$ be an endomorphism of $M$. Then, there exists $n \in \mathbb{Z}^+$ so that

$$M = \text{Ker}(\phi^n) \oplus \text{Im}(\phi^n).$$

\textbf{Proof.} Consider the chains of submodules,

$$\text{Ker}(\phi) \subseteq \text{Ker}(\phi^2) \subseteq \cdots$$

and

$$\text{Im}(\phi) \supseteq \text{Im}(\phi^2) \supseteq \cdots$$

Since $M$ in Noetherian, the first chain stabilizes, say at $\text{Ker}(\phi^n)$. Since $M$ is Artinian, the second chain stabilizes, say at $\text{Im}(\phi^m)$. Without loss of generality, we can assume that $n = m$. Then, if $x \in \text{Ker}(\phi^n) \cap \text{Im}(\phi^n)$, there exists $y \in M$ so that $\phi^n(y) = x$. Then, $0 = \phi^n(x) = \phi^n(\phi^n(y)) = \phi^{2n}(y)$ so that $y \in \text{Ker}(\phi^{2n})$. But, $\text{Ker}(\phi^{2n}) = \text{Ker}(\phi^n)$ and we have that $y \in \text{Ker}(\phi^n)$, which implies that $x = \phi^n(y) = 0$. So, $\text{Ker}(\phi^n) \cap \text{Im}(\phi^n) = \{0\}$. To conclude, we must show that $M$ is contained in the sum. Let $x \in M$. Since $\text{Im}(\phi^n) = \text{Im}(\phi^{2n})$, there exists $y \in M$ so that $\phi^n(x) = \phi^{2n}(y)$. Then, $\phi^n(x - \phi^n(y)) = 0$ so that $x - \phi^n(y) \in \text{Ker}(\phi^n)$. We can therefore write $x$ as $x = x - \phi^n(y) + \phi^n(y)$, where $x - \phi^n(y) \in \text{Ker}(\phi^n)$ and $y \in \text{Im}(\phi^n)$.

As a consequence, any endomorphism of an indecomposable module $M$ is either bijective or nilpotent.

\textbf{Corollary 2.} Let $M$ be a finitely generated indecomposable module over an Artinian ring $R$ and let $\phi$ be an endomorphism of $M$. Then, $\phi$ is either bijective or nilpotent.
**Proof.** By Fitting’s Lemma, there exists \( n \) so that

\[
M = \ker(\phi^n) \oplus \im(\phi^n).
\]

Since \( M \) is indecomposable, either \( \ker(\phi^n) = 0 \) and \( \im(\phi^n) = M \) or \( \ker(\phi^n) = M \) and \( \im(\phi^n) = 0 \). In this first case, since \( \ker(\phi) \subseteq \ker(\phi^n) = 0 \), \( \phi \) is injective. Since \( M = \im(\phi^n) \subseteq \im(\phi) \), \( \phi \) is surjective. In the second case, by definition we have that \( \phi \) is nilpotent.

Recall that if every element of \( \text{End}(M) \) is either invertible or nilpotent, we say that \( \text{End}(M) \) is local. Thus,

**Lemma 5.** A module \( M \) is indecomposable if and only if \( \text{End}(M) \) is local.

**Proof.** If \( M \) is indecomposable, then the corollary above show that \( \text{End}(M) \) is local. If \( M \) is not indecomposable, then \( M = N \oplus N' \). The projection onto \( N \) and the projection onto \( N' \) are two orthogonal idempotents, which are therefore not invertible and not nilpotent. Thus, \( \text{End}(M) \) is not local.

We next show that an indecomposable projective module gives a simple module.

**Proposition 6.** Let \( P \) be an indecomposable projective module. Then, \( P/\text{rad}(P) \) is simple.

**Proof.** Since \( P \) is indecomposable, \( \text{End}(P) \) is local. Using the projectivity of \( P \), we will show that any endomorphism of \( P/\text{rad}(P) \) can be lifted to an endomorphism of \( P \). Indeed, if \( \phi : P/\text{rad}(P) \to P/\text{rad}(P) \) is an endomorphism, using the surjectivity of the natural projection map \( \pi : P \to P/\text{rad}(P) \) and the projectivity of \( P \), there exists a map \( \tilde{\phi} : P \to P \) so that \( \pi \circ \tilde{\phi} = \phi \circ \pi \). Therefore, if \( \text{End}(P) \) is local, also \( \text{End}(P/\text{rad}(P)) \) is local. So, \( P/\text{rad}(P) \) is indecomposable. Also recall that \( P/\text{rad}(P) \) is semisimple. But, an indecomposable semisimple module is simple and we have the result.

The next step is to show that every simple module gives an indecomposable projective module. We first prove a preliminary lemma.

**Lemma 6.** If a projective indecomposable module has a maximal submodule, there is only one maximal submodule up to isomorphism.

**Proof.** Suppose that \( M \) and \( N \) are two maximal submodules of \( P \). We have the following short exact sequences.

\[
0 \to M \overset{\iota_M}{\to} P \overset{\pi_M}{\to} P/M \to 0
\]

\[
0 \to N \overset{\iota_N}{\to} P \overset{\pi_N}{\to} P/N \to 0
\]

Consider the composition \( \pi_N \circ \iota_M \). Since \( N \) is maximal, \( P/N \) is simple, and therefore \( \im(\pi_N \circ \iota_M) = 0 \) or \( \im(\pi_N \circ \iota_M) = P/N \). If \( \im(\pi_N \circ \iota_M) = P/N \), then \( \pi_N \circ \iota_M \) is surjective and, using the projectivity of \( P \), there exists a map \( \phi : P \to M \) so that the following diagram commutes.

\[
\begin{array}{ccc}
M & \overset{\pi_N \circ \iota_M}{\longrightarrow} & P/N \\
\downarrow{\phi} & & \downarrow{\pi_N} \\
0 & & 0
\end{array}
\]

Then, \( \iota_M \circ \phi \) is an endomorphism of \( P \) and is therefore invertible or nilpotent. If this composition is invertible, then \( \iota_M \circ \phi \) is surjective, which shows that \( M = P \), contradicting the maximality of \( M \). If this composition is nilpotent, then \( (\iota_M \circ \phi)^n = 0 \) for some \( n \). The commutativity of the above diagram gives that \( \pi_N = \pi_N \circ \iota_M \circ \phi \). By substituting recursively, we get that \( \pi_N = \pi_N \circ (\iota \circ \phi)^n = 0 \), contradicting that \( \pi_N \) is surjective. Therefore, \( \im(\pi_N \circ \iota_M) = 0 \). Then, \( \pi_N(\iota_M(m)) = 0 \) shows that \( M \subseteq \ker(\pi_N) = N \), which implies that \( M = N \) since \( M \) is maximal.

**Proposition 7.** Let \( S \) be a simple module. Then, \( S \cong P/\text{rad}(P) \) for some indecomposable projective module \( P \).
Proof. Since $S$ is finitely generated, $S$ is the image of a free module $M$ under a module homomorphism, say $\phi$. Decompose $M$ into a direct sum of indecomposable modules. Pick a direct summand $P$ of $M$ surjecting onto $S$ with minimal rank among all summands surjecting onto $S$. Since $P$ is a direct summand of a free module, $P$ is projective. To see that $P$ is indecomposable, suppose that $P = P' \oplus P''$. Then, since $\phi(P) = S$, also $\phi(P') \oplus \phi(P'') = S$. Since $S$ is simple, one of the direct summands is all of $S$; suppose that $\phi(P') = S$. Then, since $P$ is minimal, $P = P'$. Thus, $P$ is indecomposable. Being the homomorphic image of $P$, $S$ is isomorphic to some quotient of $P$. Since $S$ is simple, this quotient must be a quotient of $P$ by a maximal submodule. Since $P$ is projective and indecomposable, $P$ has a unique maximal submodule, and by definition of the radical, $\text{rad}(P)$ is equal to the maximal submodule. Thus, $P/\text{rad}(P) \cong S$. \qed

Proposition 8. If $P$ and $P'$ are indecomposable projective modules satisfying $P/\text{rad}(P) \cong P'/\text{rad}(P')$, then $P \cong P'$.

Proof. Let $\overline{\phi}$ be the isomorphism between the quotients. Since $P$ is projective there exists $\phi : P \to P'$ so that the following diagram commutes.

$$
\begin{array}{ccc}
P & \xrightarrow{\phi} & P' \\
\downarrow{\pi_P} & & \downarrow{\pi_{P'}} \\
P/\text{rad}(P) & \xrightarrow{\overline{\phi}} & P'/\text{rad}(P')
\end{array}
$$

Since $P'$ is projective, there exists $\psi : P' \to P$ so that $\phi \circ \psi = 1_{P'}$. Since the identity is bijective, this shows that $\phi$ is surjective and that $\psi$ is injective. To see that $\phi$ is injective, consider the following short exact sequences.

$$0 \longrightarrow \text{Ker}(\phi) \overset{\iota}{\longrightarrow} P \overset{\phi}{\longrightarrow} Q \longrightarrow 0$$

and

$$0 \longrightarrow \text{Ker}(\psi) \overset{\epsilon}{\longrightarrow} Q \overset{\psi}{\longrightarrow} \text{Im}(\psi) \longrightarrow 0$$

Since $Q$ and $\text{Im}(\psi)$ are projective, these short exact sequence split and we get that

$P \cong \text{Ker}(\phi) \oplus Q$

and

$Q \cong \text{Ker}(\psi) \oplus \text{Im}(\psi)$.

Since $\psi$ is injective, $\text{Ker}(\psi) = 0$ so that

$P \cong \text{Ker}(\phi) \oplus \text{Im}(\psi)$.

Since $\psi$ is nonzero, $\text{Im}(\psi) \neq 0$. Therefore, since $P$ is indecomposable, $\text{Ker}(\phi) = 0$ and we have that $\phi$ is injective. Thus, $\phi$ is an isomorphism and we have that $P \cong P'$. \qed

Conclusion

Theorem 1. The finitely generated projective indecomposable $kG$ modules are in bijection with the simple $kG$ modules.

Proof. Let $\mathcal{P}$ denote the set of isomorphism classes of projective indecomposable modules and let $\mathcal{S}$ denote the set of isomorphism classes of simple modules. Define

$$\phi : \mathcal{P} \to \mathcal{S}$$

$$P \mapsto P/\text{rad}(P)$$

This map is surjective by Proposition 7. This map is injective by Proposition 8. \qed
The setting to which we want to apply these ideas is the following. Let $G$ be a finite group. Let $k$ be a field and consider $P_k(G)$, the Grothendieck group with generators $[P]$ for $P$ a finitely generated projective $kG$-module and relations given by short exact sequences. Proposition 4 shows that the indecomposable projective modules span $P_k(G)$ and Proposition 5 shows that these elements form a basis for $R_k(G)$. Note that Propositions 4 and 5 are stated for a finitely generated module over an Artinian ring. In our case, $P$ is a finitely generated $kG$-module and $kG$ is Artinian since it is an algebra over a field $k$, and the propositions apply.

However, we can get away with less assumptions. More generally, Proposition 4, and hence Proposition 5, holds for any module of finite length over an arbitrary ring. Recall that a module has finite length if and only if it is both Noetherian and Artinian.

We can also ask if we can weaken the assumptions in Lemma 4. Again, this Lemma holds for any finite length module over an arbitrary ring. Since Corollary 2 only relies on Lemma 4, we can see that this Corollary holds for any finite length module over an arbitrary ring.

We therefore conclude that Theorem 1 holds for the following case. The finite length projective indecomposable modules over an arbitrary ring $R$ are in bijection with the simple $R$ modules. Also, if we consider the Grothendieck group with generators $[P]$ for $P$ a finite length projective module over an arbitrary ring $R$, we can conclude that the indecomposable projective $R$ modules form a basis for the Grothendieck group. Again, we are concerned with finitely generated $kG$ modules, but the situation holds in a more general setting.

Finally, we consider the field $k$. If char$(k) = 0$, then $kG$ is semisimple. So, any $kG$ module is semisimple and therefore, rad$(M) = 0$ for any $kG$ module. In this case, the bijection $P \leftrightarrow P/\text{rad}(P)$ just sends $P$ to $P$. Note that an indecomposable semisimple module is simple, so this map makes sense. Similarly, if char$(k) \nmid |G|$, then $kG$ is semisimple and we have the same results. The only interesting case is when char$(k) \mid |G|$. Then, $kG$ might not be semisimple and the ideas above have non trivial implications. If $kG$ is semisimple, then the indecomposable modules are the simple modules. If not, then there might be more indecomposable modules than there are simple modules. Therefore, the semisimplicity of the ring being considered is important. If $kG$ is semisimple, a basis for $P_k(G)$ is just given by the simple modules, but if $kG$ is not semisimple, then a basis for $P_k(G)$ is given by the projective indecomposable modules.

References


