# Nakayama's Lemma

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In this note, we state and prove Nakayama's Lemma. We also discuss the various corollaries and versions of Nakayama's Lemma as well as some applications of the very useful result. Throughout, $A$ is an integral domain and all $A$-modules are finitely generated.

## Statement and Proof of Nakayama's Lemma

Nakayama's Lemma gives a criterion for a module to be zero.

**Proposition.** Let $A$ be a ring, $\mathfrak{a}$ an ideal contained in all maximal ideals of $A$, and $M$ a finitely generated $A$-module. If $\mathfrak{a}M = M$, then $M = 0$.

*Proof.* A reference for the proof is [1, pg. 8]. We proceed by induction on the number of generators of $M$. For the base case, suppose that $M$ is cyclic, generated by $m_1$. The hypothesis that $\mathfrak{a}M = M$ implies that any $m \in M$ can be written as a finite sum of elements of the form $a_i m_i$, with $a_i \in \mathfrak{a}$ and $m_i \in M$. Since $\mathfrak{a}$ is an ideal of $A$ and $m_1$ generates $M$, we can see that $\mathfrak{a}M = M$ implies that any $m \in M$ can be written as a product $a_1 m_1$, with $a_1 \in \mathfrak{a}$ and $m_1 \in M$ being the generator of $M$. In particular, $m_1 = a_1 m_1$ for some $a_1 \in \mathfrak{a}$. Then, $(1 - a_1)m_1 = 0$. If $1 - a_1$ was not a unit, then $1 - a_1$ would lie in a maximal ideal of $A$. Since $a_1 \in \mathfrak{a}$ and $\mathfrak{a}$ is contained in all maximal ideals of $A$, this statement would imply that 1 lies in a maximal ideal, which is a contradiction. So, $1 - a_1$ is a unit. Multiplying $(1 - a_1)m_1 = 0$ by the inverse of $1 - a_1$, we have that $m_1 = 0$ and hence $M = 0$. Now, suppose that $M$ is generated by $\{m_1, \dots, m_n\}$. Then, since $\mathfrak{a}M = M$, we can write

$$m_1 = a_1 m_1 + \cdots + a_n m_n,$$

for $a_i \in \mathfrak{a}$. Then,

$$(1 - a_1)m_1 = a_2 m_2 + \cdots + a_n m_n.$$

Since $a_1$ is in all maximal ideals of $A$, $1 - a_1$ is a unit and we have that

$$m_1 = (1 - a_1)^{-1}(a_2 m_2 + \cdots + a_n m_n)$$

so that $m_1$ lies in a module generated by $n-1$ elements and therefore $M$ has $n-1$ generators. By induction, $M = 0$. □

Note that $\mathrm{rad}(A)$ is an ideal satisfying the conditions of the above proposition. Therefore, if $M$ is a finitely generated $A$-module and $\mathrm{rad}(A)M = M$, then $M = 0$.

## Corollaries

Restricting to local rings, we have the following corollaries.

**Corollary.** Let $A$ be a local ring with unique maximal ideal $\mathfrak{m}$. Let $M$ be a finitely generated $A$-module and let $N$ be a submodule. If $M = N + \mathfrak{m}M$, then $M = N$.

*Proof.* Consider the quotient module $M/N$ and note that $\mathfrak{m}$ is the unique maximal ideal of $A$ and hence contained in all maximal ideals of $A$. Since $M = N + \mathfrak{m}M$, $M/N = (N + \mathfrak{m}M)/N = \mathfrak{m}M/N = \mathfrak{m}(M/N)$. By Nakayama's Lemma, $M/N = 0$. Thus, $M = N$. □

**Corollary.** Let $A$ be a local ring with unique maximal ideal $\mathfrak{m}$ and let $M$ be a finitely generated $A$-module. If $m_1, \dots, m_n$ are generators for $M \mod \mathfrak{m}M$, then they are generators for $M$.

*Proof.* Let $N$ be the submodule of $M$ generated by $m_1, \dots, m_n$. Then, $M = N + \mathfrak{m}M$. By the previous corollary, we have that $M = N$. Thus, $M$ is generated by $m_1, \dots, m_n$. □

Applications and Consequences of Nakayama's Lemma

Nakayama's Lemma and the corollaries above have far reaching consequences and are used frequently in commutative ring theory. We next take a look at some of the applications of Nakayama's Lemma.

Projective modules generalize free modules. They posses most of the qualities of free modules, but do not necessarily have a basis. Recall that every free module is projective, but not every projective module is free. However, over a local ring, every finitely generated projective module is free. The following proposition uses Nakayama's Lemma to prove this statement.

**Proposition.** Let $A$ be a local ring with unique maximal ideal $\mathfrak{m}$ and let $P$ be a finitely generated projective $A$-module. Then, $P$ is free. More precisely, if $x_1, \ldots, x_n$ are elements of $P$ so that the residue classes $\overline{x}_1, \ldots, \overline{x}_n$ are a basis of $P/\mathfrak{m}P$ over $A/\mathfrak{m}$, then $x_1, \ldots, x_n$ are a basis of $P$ over $A$. If $x_1, \ldots, x_r$ are so that $\overline{x}_1, \ldots, \overline{x}_r$ are linearly independent over $A/\mathfrak{m}$, then they can be completed to a basis of $P$ over $A$.

*Proof.* First note that $P/\mathfrak{m}P$ is a module, and hence a vector space, over $A/\mathfrak{m}$. The action of $A/\mathfrak{m}$ on $P/\mathfrak{m}P$ is given by $(a+\mathfrak{m}).(p+\mathfrak{m}P) = ap + \mathfrak{m}P$. This action is well defined since if $a \in \mathfrak{m}$, then $ap \in \mathfrak{m}P$.

To show that $P$ is free, let $F$ be a free module with basis $e_1, \ldots, e_n$ and let $f : F \to P$ be the homomorphism defined by $e_i \mapsto x_i$. By a previous corollary, since $x_1, \ldots, x_n$ are generators for $P \mod \mathfrak{m}P$, they are generators for $P$. Therefore, $x_1, \ldots, x_n$ generate $P$ and $f$ is surjective. Since $P$ is projective, there exists $g : P \to F$ so that $f \circ g = 1_P$. Then, $F \cong \text{Ker}(f) \oplus P$. To conclude that $f$ is an isomorphism, we will use Nakayama's Lemma to show that $\text{Ker}(f) = 0$. If $\sum a_i e_i \in \text{Ker}(f)$, then $\sum a_i x_i = 0$. Then, $\sum a_i x_i + \mathfrak{m}P = 0 + \mathfrak{m}P$, or equivalently, $\sum (a_i\mathfrak{m}).(x_i + \mathfrak{m}P) = 0 + \mathfrak{m}P$. Since $\overline{x}_i$ form a basis for $P/\mathfrak{m}P$ over $A/\mathfrak{m}$, the previous statement shows that $a_i \in \mathfrak{m}$. So, $\sum a_i e_i \in \mathfrak{m}F$. Then, $\text{Ker}(f) \subseteq \mathfrak{m}F = \mathfrak{m}\text{Ker}(f) \oplus \mathfrak{m}P$. Therefore, $\mathfrak{m}\text{Ker}(f) = \text{Ker}(f)$ and since $\text{Ker}(f)$ is a direct summand of a finitely generated module, $\text{Ker}(f)$ is finitely generated. By Nakayama's Lemma, $\text{Ker}(f)$ and $f$ is an isomorphism. Thus, $F \cong P$ and $P$ is free.

This proof shows that $x_1, \ldots, x_n$ are a basis of $P$ over $A$. We have shown that $x_1, \ldots, x_n$ generated $P$. Since $f$ is an isomorphism and the $e_i$ are linearly independent, also $f(e_i) = x_i$ are linearly independent.

For the last statement, extend $\overline{x}_1, \ldots, \overline{x}_r$ to a basis, $\overline{x}_1, \ldots, \overline{x}_n$ of $P/\mathfrak{m}P$ over $A/\mathfrak{p}$. By the previous statements, $x_1, \ldots, x_n$ are a basis of $P$ over $A$. $\square$

Nakayama's Lemma also yields nice results for homomorphisms and induced homomorphisms on quotient modules.

**Proposition.** Let $A$ be a local ring with unique maximal ideal $\mathfrak{m}$. Let $f : M \to N$ be an $A$-module homomorphism of finitely generated $A$-modules. Then,

i. $f$ induces a homomorphism $\overline{f} : M/\mathfrak{m}M \to N/\mathfrak{m}N$

ii. $f$ is surjective if and only if $\overline{f}$ is surjective.

iii. If $M$ and $N$ are free and if $\overline{f}$ is injective, then $f$ is injective.

*Proof.* Define

$$\begin{aligned} \overline{f} : M/\mathfrak{m}M &\to N/\mathfrak{n}N \\ x + \mathfrak{m}M &\mapsto f(x) + \mathfrak{m}N \end{aligned}$$

and note that $\overline{f}$ is well defined since if $x \in \mathfrak{m}M$, $f(x) \in \mathfrak{m}N$ as $f$ is $A$-linear. This definition of $\overline{f}$ yields the following commutative diagram.

$$\begin{array}{ccc} M & \xrightarrow{\ f\ } & N \\ \big\downarrow{\scriptstyle \pi_M} & & \big\downarrow{\scriptstyle \pi_N} \\ M/\mathfrak{m}M & \xrightarrow{\ \overline{f}\ } & N/\mathfrak{m}N \end{array}$$

By definition, if $f$ is surjective then $\overline{f}$ is surjective. Suppose that $\overline{f}$ is surjective. Consider the submodule $\mathrm{Im}(f)$ of $N$. Since $\overline{f}$ is surjective, $N = \mathrm{Im}(f) + \mathfrak{m}N$. If $n \in N$, then there exists $m \in M$ so that $\overline{f}(m + \mathfrak{m}M) = n + \mathfrak{m}N$. By definition of $\overline{f}$, $f(m) + \mathfrak{m}N = n + \mathfrak{m}N$ and $n - f(m) \in \mathfrak{m}N$. Then, $n = f(m) + n - f(m) \in \mathrm{Im}(f) + \mathfrak{m}N$. By a corollary to Nakayama's Lemma, this statement shows that $N = \mathrm{Im}(f)$ and hence $f$ is surjective. Now, assume that $M$ and $N$ are free and that $\overline{f}$ is injective. Let $x_1, \ldots, x_n$ be elements of $M$ so that $\overline{x}_1, \ldots, \overline{x}_n$ is a basis of $M/\mathfrak{m}M$. Then, since $M$ is free, by a previous proposition, $x_1, \ldots, x_n$ is a basis of $M$ over $A$. Suppose that $f(a_1x_1 + \cdots + a_nx_n) = 0$. By definition of $\overline{f}$, we also have that $\overline{f}(\overline{a}_1\overline{x}_1 + \cdots + \overline{a}_n\overline{x}_n) = 0$ in $N/\mathfrak{m}N$. Since $\overline{f}$ is injecitve, $\overline{a}_1\overline{x}_1 + \cdots + \overline{a}_n\overline{x}_n = 0$. Since the $\overline{x}_1, \ldots, \overline{x}_n$ form a basis, $\overline{a}_i = 0$ for all $1 \leq i \leq n$. That is, $a_i \in \mathfrak{m}$. We next show that $\mathrm{Ker}(f) \subseteq \mathfrak{m}\mathrm{Ker}(f)$. We have that $\mathrm{Ker}(f) \subseteq \mathfrak{m}M$. We can simplify this proof if we assume that $\overline{f}$ is surjective and therefore, by a previous part of this proof, that $f$ is surjective. We have that following short exact sequence

$$0 \to \mathrm{Ker}(f) \to M \to N \to 0$$

which splits since $N$ is free. So, $M \cong \mathrm{Ker}(f) \oplus N$. Then,

$$\mathrm{Ker}(f) \subseteq \mathfrak{m}M \cong \mathrm{Ker}(f) \oplus N$$

so that $\mathrm{Ker}(f) \subseteq \mathfrak{m}\mathrm{Ker}(f)$. By Nakayama's Lemma, $\mathrm{Ker}(f) = 0$ and $f$ is injective. Now, if $\overline{f}$ is not surjective, we proceed as follows. Recall that we have a basis $\{x_1, \ldots, x_n\}$ of $M$ over $A$ and a basis $\{\overline{x}_1, \ldots, \overline{x}_n\}$ of $M/\mathfrak{m}M$ over $A/\mathfrak{m}$. The images $\{\overline{f}(\overline{x}_1), \ldots, \overline{f}(\overline{x}_n)\}$ generate $\mathrm{Im}(\overline{f})$ as a submodule of $N/\mathfrak{m}N$. Since $\overline{f}$ is injective, this set is actually a basis. If

$$\overline{a}_1\overline{f}(\overline{x}_1) + \cdots + \overline{a}_n\overline{f}(\overline{x}_n) = 0,$$

then by definition of $\overline{f}$, we have that $f(\overline{a}_1\overline{x}_1 + \cdots + \overline{a}_n\overline{x}_n) = 0$. Therefore, $\overline{a}_i = 0$ for all $i$ since the $\overline{x}_i$ are a basis for $M/\mathfrak{m}M$ over $A/\mathfrak{m}$. Extend $\{\overline{f}(\overline{x}_1), \ldots, \overline{f}(\overline{x}_n)\}$ to a basis $\{\overline{f}(\overline{x}_1), \ldots, \overline{f}(\overline{x}_n), \overline{y}_1, \ldots, \overline{y}_m\}$ of $N/\mathfrak{m}N$ over $A/\mathfrak{m}$. Since $N$ is free, this basis lifts to a basis of $N$ over $A$. Note that $\overline{f}(\overline{x}_i) = \overline{f}(x_i + \mathfrak{m}M) = f(x_i) + \mathfrak{m}N$ so that the lifted basis of $N$ over $A$ is $\{f(x_1), \ldots, f(x_n), y_1, \ldots, y_m\}$. In particular, the $f(x_i)$ are linearly independent so that if

$$a_1f(x_1) + \cdots + a_nf(x_n) = 0,$$

then $a_i = 0$ for all $i$. Since $f$ is an $A$-module homomorphism, this statement is equivalent to the statement that if $f(a_1x_1 + \cdots + a_nx_n) = 0$, then $a_i = 0$ for all $i$. Since the $x_i$ form a basis for $M$ over $A$, we have our result. Thus, $f$ is injective. □

This previous proposition is related to another claim regarding induced homomorphisms used in representation theory and stated in [2]. We simplify the situation and ignore the group ring structure.

**Proposition.** Let $A$ be a local ring with unique maximal ideal $\mathfrak{m}$. If $P$ and $P'$ are finitely generated projective $A$-modules, then $P \cong P'$ if and only if $P/\mathfrak{m}P' \cong P/\mathfrak{m}P'$.

*Proof.* This proposition follows from the previous proposition since any finitely generated projective module over a local ring is free. □

The following is just a reformulation of Nakayama's Lemma using the tensor product.

**Lemma.** Let $A$ be a local ring with unique maximal ideal $\mathfrak{m}$ and let $M$ be a finitely generated $A$-module. If $A/\mathfrak{m} \otimes_R M = 0$, then $M = 0$.

*Proof.* This tensor product formulation of Nakayama's Lemma just relies on the isomorphism $M/\mathfrak{m}M \cong A/\mathfrak{m} \otimes_R M$. We have that $M = \mathfrak{m}M$ implies $M/\mathfrak{m}M = 0$ implies $k \otimes_R M = 0$. The result then follows from the original form of Nakayama's Lemma. □

Recall that any surjective vector space homomorphism between finite dimensional vector spaces of the same dimension is also an isomorphism. This result carries over to finitely generated modules if we restrict to endomorphisms.

**Proposition.** If $M$ is a finitely generated module any surjective endomorphism of $M$ is also an isomorphism.

*Proof.* This proof will use a slightly different formulation of Nakayama's Lemma. If $I$ is an ideal of $A$ and $IM = M$, then there exists $a \in A$ so that $aM = 0$ and $1 - a \in I$. Let the polynomial ring $A[x]$ act on $M$ by $g(x).m = g(f)(m)$. That is, take a polynomial $g(x)$, evaluate at $f$, and then evaluate the resulting function on $m$. Consider the ideal of $A[x]$ generated by $x$, that is, $(x)$. Note that $x.m = f(m)$. Since $f$ is surjective, $(x)M = M$. By Nakayama's Lemma, there exists $g(x) \in A[x]$ so that $g(x)M = 0$ and $1 - g(x) \in (x)$. Write $1 - g(x) = xh(x)$ for some $h(x) \in A[x]$. Then, $(1 - xh(x))M = 0$ implies that if $m \in \text{Ker}(f)$, we have that $(1 - xh(x)).m = 0$. That is, $m - h(x).f(m) = 0$. Since $m \in \text{Ker}(f)$, $f(m) = 0$ and the previous equality becomes $m = 0$. Therefore, $\text{Ker}(f) = 0$. Thus, $f$ is an isomorphism. □

**Proposition.** The fraction field of an integral domain is not finitely generated

*Proof.* Let $R$ be an integral domain with field of fractions $F$. Suppose, for the sake of contradiction, that the $R$-module $F$ is finitely generated. Since $\text{rad}(R)$ does not contain any units, $\text{rad}(R)F = F$. By Nakayama's Lemma, $F = 0$, which is a contradiction. Thus, $F$ is not finitely generated. □

**Proposition.** Let $A$ be a Noetherian local ring with maximal ideal $\mathfrak{m}$ and suppose that $\mathfrak{m}^{n+1} = \mathfrak{m}^n$. Then, $\mathfrak{m}^n = 0$.

*Proof.* Since $A$ is Noetherian, the maximal ideal $\mathfrak{m}$ is finitely generated. In particular, $\mathfrak{m}$ is a finitely generated $A$-module. The assumption implies that $\mathfrak{m}\mathfrak{m}^n = \mathfrak{m}^n$. Thus, $\mathfrak{m}^n = 0$. □

**Proposition.** Let $A$ be a Noetherian integral domain and let $P$ be a prime ideal. Then, the powers of $P$ are distinct.

*Proof.* Since $A$ is Noetherian, $P$ is a finitely generated $A$-module. Localize at $P$ and note that $A_P$ is a local ring with maximal ideal $PA_P$, which is also a finitely generated $A_P$ module. We proceed using contradiction. Since $P$ is prime, it suffices to assume that $P^{n+1} = P^n$ for some $n$. Then, $P^{n+1}A_P = P^nA_P$ and by the previous proposition, $P^nA_P = 0$ in $A_P$. But then, $P^n = 0$. Since $P$ is prime, $P^n = 0$ implies that $P = 0$, which is a contradiction. Thus, the powers of $P$ are distinct. □

**Proposition.** Let $A$ be a local Noetherian domain with fraction field $K$ and residue field $k$ and let $M$ be a finitely generated $A$-module. Then, $M$ is free if and only if

$$\dim_K(K \otimes_A M) = \dim_k(k \otimes_A M).$$

*Proof.* Suppose that $M$ is a free $A$-module of rank $n$ and that $\{m_1, \ldots, m_n\}$ is a basis for $M$ over $A$. Since $M$ is free over $A$, $K \otimes_A M$ has $K$-basis $\{1 \otimes m_1, \ldots, 1 \otimes m_n\}$. Since $M$ is free over $A$, $k \otimes_A M$ has $k$-basis $\{1 \otimes m_1, \ldots, 1 \otimes m_n\}$. Thus,

$$\dim_K(K \otimes_A M) = r = \dim_k(k \otimes_A M).$$

Conversely, suppose that the dimensions of the tensors products are equal. Since $M$ is a finitely generated $A$-module, there exists a free $A$-module $F$ and a surjective map $\phi : F \to M$. We then have the short exact sequence

$$0 \to \text{Ker}(\phi) \to F \xrightarrow{\phi} M \to 0$$

Since the localization functor is exact and the tensor product is right exact, we get the exact sequences,

$$0 \to K \otimes_A \text{Ker}(\phi) \to K \otimes_A F \xrightarrow{id \otimes \phi} K \otimes_A M \to 0$$

and

$$k \otimes_A \text{Ker}(\phi) \to k \otimes_A F \xrightarrow{id \otimes \phi} k \otimes_A M \to 0$$

From here, we can take two different routes of proof.

i. Since the modules being considered are vector spaces over fields, we can add dimensions in exact sequences. From the first sequence, we have that

$$\dim_K(K \otimes_A F) = \dim_K(K \otimes_A M) + \dim_K(K \otimes_A \operatorname{Ker}(\phi)).$$

Since $F$ is free over $A$, $\dim_K(K \otimes_A F) = \dim_k(k \otimes_A F)$. By hypothesis, we then have that

$$\dim_k(k \otimes_A F) = \dim_k(k \otimes_A M) + \dim_K(K \otimes_A \operatorname{Ker}(\phi))$$

which shows that $\dim_k(k \otimes_A \operatorname{Ker}(\phi)) = \dim_K(K \otimes_A \operatorname{Ker}(\phi))$. But, $\dim_K(K \otimes_A \operatorname{Ker}(\phi)) = \dim_K(\operatorname{Im}(id \otimes \iota)) = \dim_k(\operatorname{Im}(id \otimes \iota))$, which implies that

$$0 \to k \otimes_A \operatorname{Ker}(\phi) \to k \otimes_A F \xrightarrow{id \otimes \phi} k \otimes_A M \to 0$$

is exact. Now, choose $F$ so that $\dim_k(k \otimes_A F) = \dim_k(k \otimes_A M)$. Then, $id \otimes \phi$ is a surjective map between finite dimensional vector spaces of the same dimension. Thus, $id \otimes \phi$ is an isomorphism and $k \otimes_A \operatorname{Ker}(\phi) = 0$. By Nakayama's Lemma, $\operatorname{Ker}(\phi) = 0$ and $\phi$ is an isomorphism. Thus, $M$ is free.

ii. Suppose that $\dim_k(k \otimes_A M) = r = \dim_K(\otimes_A M)$. By Nakayama's Lemma, take basis for $k \otimes_A M$ over $k$ and lift to a generating set for $M$ over $A$. Then, we can choose $F$ to be free of rank $r$. Then, $\dim_K(K \otimes_A F) = r = \dim_K(K \otimes_A M)$, which implies that $K \otimes_A \operatorname{Ker}(\phi) = 0$. Since $\operatorname{Ker}(\phi)$ is a submodule of a free module, $\operatorname{Ker}(\phi)$ is torsion free. Therefore, $\operatorname{Ker}(\phi)$. Thus, $A$ is free.

□

Historical Note

The lemma in is current form was introduced in 1951 by Tadasi Nakayama in the article "A remark on finitely generated modules" Nagoya Mathematical Journal, 3: pg. 139-140. However, it was first discovered by Wolfgang Krull in the specific case of ideals of a commutative ring. In 1951, Goro Azumaya also discovered this lemma. In fact, Nakayama's Lemma is also known as the Krull-Azumaya Theorem. According to Matsumura, in *Commutative Algebra*, "This simple but important lemma is due to T. Nakayama, G. Azumaya and W. Krull. Priority is obscure, and although it is usually called the Lemma of Nakayama, late Prof. Nakayama did not like the name." There are also other versions of this lemma. There is a noncommutative Nakayama's Lemma as well as a graded Nakayama's Lemma.

# References

[1] Serge Lang. *Algebraic Number Theory.* Springer-Verlag, New York, second edition, 1994.

[2] Jean-Pierre Serre. *Linear Representations of Finite Groups.* Springer-Verlag, New York, 1977.