Verifying Bit-vector Invertibility Conditions in Coq (Extended Abstract)

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Bit-vectors

Bit-vectors

- Bit-vectors: Fixed-width bit sequences

    10101

    $a \in BV_5$

Bit-vectors

- Bit-vectors: Fixed-width bit sequences

  10101

  $a \in BV_5$

- Bit-vector operations

  $a + b$

  $a <_u b$

  …

Introduction

Bit-vectors have many applications:

- Hardware circuit analysis [Gupta et al., 1993]
- Bounded model checking [Armando et al., 2006]
- Symbolic execution [Cadar et al., 2006]
- ...

Introduction

- Many applications require quantified bit-vector formulas
- Some SMT solvers use quantifier-instantiation to solve quantified formulas
- *Invertibility conditions* are a useful meta-construct for a quantifier-instantiation technique [Niemetz et al., CAV 2018]

Invertibility Conditions

An *invertibility condition* for a variable x in a bit-vector literal

$$\ell\,[\,x\,,\;s\,,\;t\,]$$

is a formula

$$IC\,[\,s\,,\;t\,]$$

s.t. the following *invertibility equivalence* is valid in the theory of bit-vectors:

$$\forall s.\ \forall t.\ IC[s,t] \iff \exists x.\ \ell[x,s,t]$$

where s, t, x : BV_n

Invertibility Conditions: Example

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- Inversion of bit-vector addition is unconditional

$$\exists x.\ x + s = t \iff \top$$

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The *inverse* is $x = t - s$

- Inversion of bit-wise conjunction is conditional

$$\exists x.\ x \;\&\; s = t \iff t \;\&\; s = t$$

Motivation

This technique [Niemetz et al., CAV 2018] requires the equivalences to be true independent of bit-width

Proofs of these equivalences are required for the soundness of the technique

…and the solvers that use it

Previous Work

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[Niemetz et al., CAV 2018]

- generated 162 invertibility equivalences
- proved them using SMT-solvers for bit-widths up to 65

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- generated 162 invertibility equivalences
- proved them using SMT-solvers for bit-widths up to 65

[Niemetz et al., CADE 2019]

- encoded the equivalences in theories supported by SMT-solvers
- verified equivalences for parametric bit-widths
- approach succeeded on under 75% of the equivalences

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1. Formalized a representative subset of the 162 invertibility equivalences in Coq

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2. Extended a Coq bit-vector library to support these equivalences

3. Proved 18 of them for arbitrary bit-width

Result Summary

$\ell[x]$	$=$	$\neq$	$<_u$	$>_u$	$\leq_u$	$\geq_u$
$-x \bowtie t$	✓✓	✓	✓	✓	✓	✓
$\sim x \bowtie t$	✓✓	✓	✓	✓	✓	✓
$x \,\&\, s \bowtie t$	✓	✓	✓	✓	✓	✓
$x \mid s \bowtie t$	✓	✓	✓	✓	✓	✓
$x \ll s \bowtie t$	✓	✓	✓	✓	✓	✓
$s \ll x \bowtie t$	✓✓	✓	✓	✓	✓	✓
$x \gg s \bowtie t$	✓✓	✓	✓	✗	✓	✓
$s \gg x \bowtie t$	✓✓	✓	✓	✓	✓	✓
$x \gg_a s \bowtie t$	✓	✓	✓	✓	✓	✓
$s \gg_a x \bowtie t$	✓✓	✓	✓	✓	✓	✓
$x + s \bowtie t$	✓✓	✓	✓	✓	✓	✓

Result Summary (SMT)

$\ell[x]$	$=$	$\neq$	$<_u$	$>_u$	$\leq_u$	$\geq_u$
$-x \bowtie t$	✓	✓	✓	✓	✓	✓
$\sim x \bowtie t$	✓	✓	✓	✓	✓	✓
$x \mathbin{\&} s \bowtie t$		✓	✓	✓	✓	✓
$x \mid s \bowtie t$		✓	✓	✓	✓	✓
$x \ll s \bowtie t$			✓		✓	
$s \ll x \bowtie t$	✓	✓	✓	✓	✓	✓
$x \gg s \bowtie t$	✓	✓	✓		✓	✓
$s \gg x \bowtie t$	✓	✓	✓	✓	✓	✓
$x \gg_a s \bowtie t$		✓	✓	✓	✓	✓
$s \gg_a x \bowtie t$	✓	✓				
$x + s \bowtie t$	✓	✓	✓	✓	✓	✓

Result Summary (Coq)

$\ell[x]$	$=$	$\neq$	$<_u$	$>_u$	$\leq_u$	$\geq_u$
$-x \bowtie t$	✓					
$\sim x \bowtie t$	✓					
$x \;\&\; s \bowtie t$	✓					
$x \mid s \bowtie t$	✓					
$x \ll s \bowtie t$	✓	✓		✓		✓
$s \ll x \bowtie t$	✓					
$x \gg s \bowtie t$	✓			✗		
$s \gg x \bowtie t$	✓					
$x \gg_a s \bowtie t$	✓					
$s \gg_a x \bowtie t$	✓		✓	✓	✓	✓
$x + s \bowtie t$	✓					

Result Summary (Both)

$\ell[x]$	$=$	$\neq$	$<_u$	$>_u$	$\leq_u$	$\geq_u$
$-x \bowtie t$	$\color{blue}{\checkmark}\color{green}{\checkmark}$	$\color{green}{\checkmark}$	$\color{green}{\checkmark}$	$\color{green}{\checkmark}$	$\color{green}{\checkmark}$	$\color{green}{\checkmark}$
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$s \gg x \bowtie t$	$\color{blue}{\checkmark}\color{green}{\checkmark}$	$\color{green}{\checkmark}$	$\color{green}{\checkmark}$	$\color{green}{\checkmark}$	$\color{green}{\checkmark}$	$\color{green}{\checkmark}$
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Bit-vector Library

Bit-vector Library

- Used a bit-vector library originally developed for SMTCoq [Ekici et al., 2017]
- SMTCoq is a Coq plugin that uses external SMT solvers to complete proof goals
- Bit-vectors are represented as lists of Booleans

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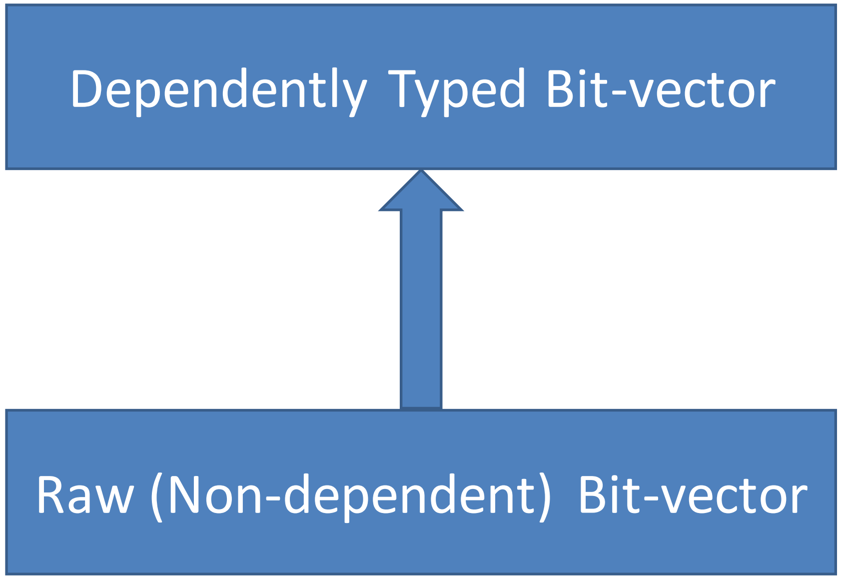
Modules



Types

bitvector : N -> Type

Functor: Raw2Bitvector

bitvector : Type
size : bitvector -> N

Bit-vector Representations

	SMTLib [Niemetz et al. 18]	Encoding [Niemetz et al. 19]	Coq Library
Bit-vector Representation:	Bit-vector of width n One sort for each n	Bit-vector of width n Translated to NIA and UF	Bit-vector of width n List of Booleans over 2 layers
Expressivity:	n cannot be symbolic	Allows quantification over n	Bit-vectors dependent over n
Verification:	Automatic proofs using SMT solvers	Automatic proofs using SMT solvers	Manual proofs in Coq
Results	All equivalences for n = 1 to 65	Verified ≈75% of equivalences	18 equivalences

Bit-vector Library

Basic signature (previous work):

$+$	addition	$\circ$	concatenation
$-$	negation	$=$	equality
$\bullet$	multiplication	$\neq$	disequality
$\&$	bit-wise conjunction	$<_u$	unsigned less than
$\mid$	bit-wise disjunction	$>_u$	unsigned greater than
$\sim$	bit-wise negation	$<_s$	signed less than
$<<$	logical left shift	$>_s$	signed greater than
$>>$	logical right shift		

Extended signature (this work):

$\leq_u$	unsigned weak less than	$\underline{<<}$	logical left shift (alt. def.)
$\geq_u$	unsigned weak greater than	$\underline{>>}$	logical right shift (alt. def.)
$>>_a$	arithmetic right shift	$\underline{>>}_a$	arithmetic right shift (alt. def.)

Original Shift Definition

```
Definition shl_one_bit (a: list bool) : list bool :=
  match a with
    | [] => []
    | _ => false :: removelast a
  end.

Fixpoint shl_n_bits (a: list bool) (n: nat): list bool :=
  match n with
    | O => a
    | S n' => shl_n_bits (shl_one_bit a) n'
  end.

Definition shl_aux (a b: list bool): list bool :=
 shl_n_bits a (list2nat_be_a b).
```

Shift Redefined

```
Definition shl_n_bits_a (a: list bool) (n: nat): list bool :=
  if (n <? length a)%nat then
    mk_list_false n ++ firstn (length a - n) a
  else
    mk_list_false (length a).

Definition bv_shl_a (a b: bitvector) : bitvector :=
  if ((@size a) =? (@size b)) then
    shl_n_bits_a a (list2nat_be_a b)
  else
    nil.
```

Invertibility Conditions Proofs

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$$\texttt{bvshr\_ugt\_rtl}: \forall n.\ \forall x, s, t : BV_n.\ (x \gg s) <_u t \ \rightarrow\ t <_u (\sim s \gg s)$$

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Challenge

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Consider bit-vectors s where $k < l$.
For $l = 4$,

s	k	l – k
0000	0	4
0001	1	3
0010	2	2
0011	3	1

$k = toNat(s)$
$l = length(s)$

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reduces to

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Proof Sketch of msb_zero

`msb_zero:` $\forall s : BV_n.\ k < l \rightarrow s[(l-1)...k] = [0...0]$

Proof Sketch:

$$\begin{array}{lcccccccc} & & l-1 & & k & k-1 & & 0 & \\ \text{s:} & [& 0 & \dots & 0 & \_ & \dots & \_ &] \end{array}$$

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$$k = \sum_{i=0}^{l-1} s[i] \cdot 2^i$$

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But $k < l$

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But $k < l$

$$= s[l-1] \cdot 2^{l-1} + \ ... \ + s[k] \cdot 2^k + s[k-1] \cdot 2^{k-1} + \ ... \ + s[1] \cdot 2^1 + s[0] \cdot 2^0$$

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0010	2	2
0011	3	1

Proof Sketch of msb_zero

`msb_zero:` $\forall s : BV_n.\ k < l \rightarrow s[(l-1)...k] = [0...0]$

Proof Sketch:

$$
\begin{array}{cccccccc}
l-1 & & & k & k-1 & & & 0 \\
\end{array}
$$

$$\text{s: } [\ 0\ \ldots\ 0\ \_\ \ldots\ \_\ ]$$

$$k = \sum_{i=0}^{l-1} s[i] \cdot 2^i$$

$$= s[l-1] \cdot 2^{l-1} + ... + s[1] \cdot 2^1 + s[0] \cdot 2^0$$

But $k < l$

$$= s[l-1] \cdot 2^{l-1} + \ \ ... \ \ + s[k] \cdot 2^k + s[k-1] \cdot 2^{k-1} + \ \ ... \ \ + s[1] \cdot 2^1 + s[0] \cdot 2^0$$

But $k < 2^k < 2^{k+1} < ... < 2^{l-1}$

For $l = 4$,

s	k	l – k
0000	0	4
0001	1	3
0010	2	2
0011	3	1

Proof Sketch of msb_zero

`msb_zero:` $\forall s : BV_n.\ k < l \rightarrow s[(l-1)...k] = [0...0]$

Proof Sketch:

$$\begin{array}{cccccccccc} & & l-1 & & k & k-1 & & 0 & \\ \mathrm{s:} & [& 0 & \ldots & 0 & \_ & \ldots & \_ &] \end{array}$$

$$k = \sum_{i=0}^{l-1} s[i] \cdot 2^i$$

$$= s[l-1] \cdot 2^{l-1} + ... + s[1] \cdot 2^1 + s[0] \cdot 2^0$$

But $k < l$

$$= s[l-1] \cdot 2^{l-1} + \ \ldots\ + s[k] \cdot 2^k + s[k-1] \cdot 2^{k-1} + \ \ldots\ + s[1] \cdot 2^1 + s[0] \cdot 2^0$$

But $k < 2^k < 2^{k+1} < ... < 2^{l-1}$

Thus, the coefficients of $2^k, ..., 2^{l-1}$ are 0.

For $l = 4$,

s	k	l – k
0000	0	4
0001	1	3
0010	2	2
0011	3	1

Conclusion

- [Niemetz et al., CAV 2018] presented 162 invertibility conditions
- [Niemetz et al., CADE 2019] verified ≈75% of the equivalences using an encoding
- We complemented [Niemetz et al., CADE 2019] in proving all but one invertibility equivalences from 66 of them
- We did this in the Coq proof assistant
- We extended the Coq bit-vector library for SMTCoq to do this

Future Work

- Integrate the extended bit-vector library into SMTCoq
- Prove the remaining invertibility equivalences
- Extend the library with $/$ $\%$ $\leq_s$ $\geq_s$
- Refactor the library for improved readability and modularity
- Consider using MathComp library for future proofs

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Useful Lemmas

```
Lemma firstn_all: ∀ (l : list A), firstn (length l) l = l.

Lemma firstn_length_le: ∀ (l : list A) (n : nat), n <= length l
-> length (firstn n l) = n.

Lemma firstn_length: ∀ (n : nat) (l : list A), length (firstn n
l) = min n (length l).

Theorem app_nil_r: ∀ (l : list A), l ++ [] = l.

Lemma app_length: ∀ (l l' : list A), length (l++l') = length l +
length l'.
```