$$\nabla f = (2x - y, -x) \qquad \boxed{\nabla f(1,2) = (0,-1)}$$

5.) Let $f(x, y) = x^2 - xy$ $\qquad \nabla f(0,2) = (-2, 0)$

$f(0,2) = 0$

[5] 5a.) Calculate the Hessian matrix of f at $(x, y) = (0, 2)$

$$Hf = \begin{pmatrix} 2 & -1 \\ -1 & 0 \end{pmatrix}$$

[5] 5b.) Find the first order Taylor polynomial for f at $(x, y) = (0, 2)$ = tangent line

$$p_1(\vec{x}) = f(\vec{a}) + Df|_{\vec{a}}(\vec{x} - \vec{a})$$

$$= 0 + (-2, 0)\begin{pmatrix} x - 0 \\ y - 2 \end{pmatrix} = -2x$$

$\cancel{p_1(x,y) =}$ $\qquad \boxed{P_1(x,y) = -2x}$

[5] 5c.) Find the second order Taylor polynomial for f at $(x, y) = (0, 2)$

$$P_2(\vec{x}) = P_1(x) + \frac{1}{2}(\vec{x} - \vec{a})^T Hf|_{\vec{a}}(\vec{x} - \vec{a}) \qquad \boxed{P_2(x,y) = x^2 - xy}$$

$$= -2x + \frac{1}{2}(x, y-2)\begin{pmatrix} 2 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y-2 \end{pmatrix}$$

$x^2 - xy$
$//$

$$= -2x + \frac{1}{2}(x, y-2)\begin{pmatrix} 2x - (y-2) \\ -x \end{pmatrix} = -2x + [2x^2 - xy + 2x - xy + 2x]\frac{1}{2}$$

[5] 5d.) Use the fact that the total differential df approximates the incremental change Δf to estimate $f(0.98, 2.1)$. $\qquad \vec{a} = (1, 2) \qquad f(1,2) = -1$

$$df = Df|_a(\vec{x} - \vec{a})$$

$$= (0, -1)\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = -\Delta y$$

$$\Delta f \sim df = -\Delta y = -(2.1 - 2) = -0.1$$

$f(0.98, 2.1)$

$\sim -1 + -0.1$

$= -1.1$

[3] 5e.) Is the value of the function near $(x, y) = (0, 2)$ more sensitive to changes in x or y?

more sensitive to changes in y

Problem 5 continued.)

[3] 5f.) Give an equation for the tangent plane to f at $(x, y) = (0, 2)$

$$p_1(x,y) = -2x$$

[4] 5g.) The critical points of f are $\underline{\quad (0,0) \quad}$

$$\nabla f = (2x-y, -x) = (0,0)$$

$$\Rightarrow x=0, y=0$$

[3] 5h.) Use the sequence of principal minors of the Hessian of f to determine the nature of the critical points (i.e, local max/min/saddle or no info).

$$Hf = \begin{pmatrix} 2 & -1 \\ -1 & 0 \end{pmatrix}$$

$$d_1 = 2 > 0$$

$$d_2 = \begin{vmatrix} 2 & -1 \\ -1 & 0 \end{vmatrix} = 0-1 < 0$$

6.) Let $F(x, y, z) = (0, xy, z)$

[5] 6a.) The divergence of $F = \underline{\quad x+1 \quad}$

$$\nabla \cdot F = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (0, xy, z)$$

$$= 0 + x + 1$$

[5] 6b.) The curl of $F = \underline{\quad (0,0,y) \quad}$

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & xy & z \end{vmatrix} = (0, 0, y)$$

velocity =
$x'(t) = (-4\sin t,\ 4\cos t,\ 3)$ | Speed $= \|x'(t)\|$
$= \sqrt{16\sin^2 t + 16\cos^2 t + 9}$
$= \sqrt{16+9} = \sqrt{25} = 5$

7.) Let $x(t) = (4cos(t), 4sin(t), 3t)$, $1 \le t \le 5$, be a path in $\mathbf{R}^3$.

[5] 7a.) Find the arclength parameter $s = s(t)$ for this path.

$$s(t) = \int_1^t 5\,du = 5u\Big|_1^t = 5t - 5$$

$$\boxed{s(t) = 5t - 5}$$

[3] 7b.) The length of this path is 20.

$$s(5) = 25 - 5 = 20$$

[5] 7c.) Express the original parameter t in terms of s and reparametrize the path $\mathbf{x}$ in terms of s. $s = 5t - 5 \Rightarrow s + 5 = 5t \Rightarrow t = \frac{s+5}{2}$

$$x(s) = \left(4\cos\left(\frac{s+5}{2}\right),\ 4\sin\left(\frac{s+5}{2}\right),\ 3\left(\frac{s+5}{2}\right)\right)$$

[5] 7d.) Determine the moving frame [$\mathbf{T}, \mathbf{N}, \mathbf{B}$]

$$T(t) = \frac{x'(t)}{\|x'(t)\|} = \boxed{\left(-\frac{4\sin t}{5},\ \frac{4\cos t}{5},\ \frac{3}{5}\right) = T(t)}$$

$$N = \frac{T'(t)}{\|T'(t)\|} = \left(\frac{-4}{5}\cos t,\ -\frac{4\sin t}{5},\ 0\right)\Big/\|T'(t)\|$$
$$= \boxed{(-\cos t,\ -\sin t,\ 0) = N(t)}$$

$$B = T \times N = \begin{vmatrix} i & j & k \\ \frac{-4\sin t}{5} & \frac{4\cos t}{5} & \frac{3}{5} \\ -\cos t & -\sin t & 0 \end{vmatrix} = \boxed{\left(\frac{3}{5}\sin t,\ -\frac{3}{5}\cos t,\ \frac{4}{5}\right) = B(t)}$$

$\mathbf{T} =$ ~~$\left(-\frac{4\sin t}{5}, 4\cos t\right.$~~ ______, $\mathbf{N} =$ ______, $\mathbf{B} =$ ______

[5] 7e.) The curvature of this path is 4/25

$$\kappa(t) = \frac{\|T'(t)\|}{s'(t)} = \frac{\sqrt{\frac{16}{25}\cos^2 t + \frac{16\sin^2 t}{25} + 0^2}}{5} = \frac{\frac{4}{5}}{5} = \frac{4}{25}$$

[5] 7f.) The torsion of this path is 3/25

$$\frac{B'(t)}{s'(t)} = \frac{\left(\frac{3}{5}\cos t,\ \frac{3}{5}\sin t,\ 0\right)}{5}$$

$$= \left(\frac{3}{25}\cos t,\ \frac{3}{25}\sin t,\ 0\right) = -\tau(-\cos t, -\sin t, 0)$$

$$\Rightarrow \tau = 3/25$$