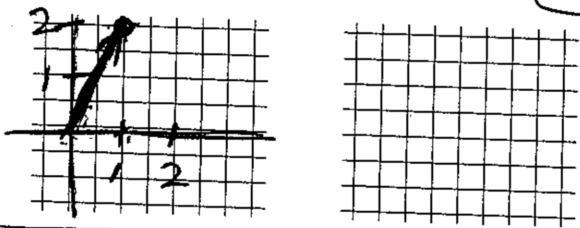
$w \overset{N}{\underset{S}{}} E$



$h: [0,\infty) \to \mathbb{R}$
$h(x) = \sqrt{x}$
is a fn

1.1 Vectors:

Let $\mathbf{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

If $\mathbf{v}$ = velocity in m/sec of an object, then the object is moving east at a rate of 1 m/sec and north at a rate of 2m/sec

1rst comp — 2nd comp

Speed of the object =

length of $v = \|v\| = \sqrt{1^2 + 2^2}$

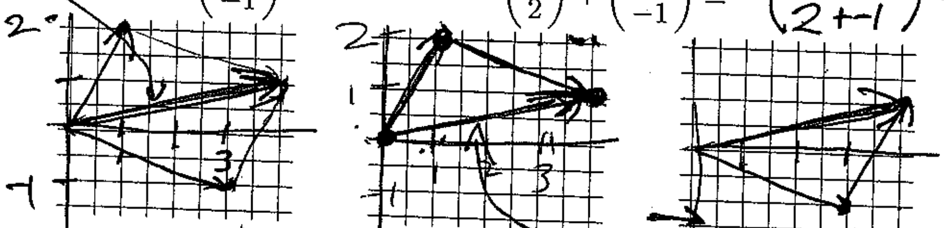
$= \sqrt{5}$ m/sec

A vector can be described by its Euclidean coordinates OR by its length and direction.

$\vec{v} + \vec{w}$ diagonal of $\vec{0}$ at $\vec{v}+\vec{w}$

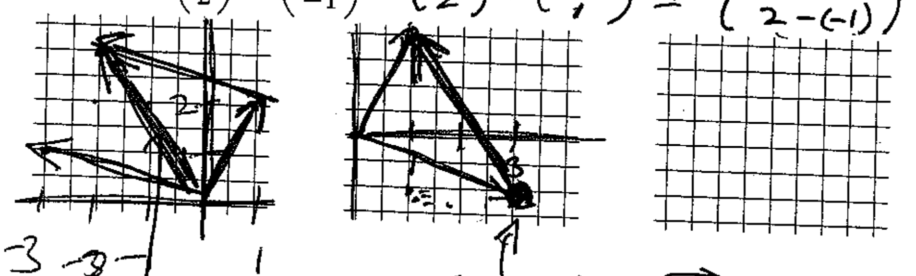
starting, ending

Let $\mathbf{w} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$. Then $\mathbf{v} + \mathbf{w} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 1+3 \\ 2+-1 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$



$\vec{v} + \vec{w}$

$v + -w$ $\quad \mathbf{v} - \mathbf{w} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} -3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1-3 \\ 2-(-1) \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$



$v - w$

$\vec{v} - \vec{w}$ is the vector starting at the point $\vec{w}$ ending at the point $\vec{v}$

domain, codomain

2.1 Let $f : X \to Y$ where $X \subset \mathbf{R}^n$, $Y \subset \mathbf{R}^m$

Graph of $f = \{(\mathbf{x}, f(\mathbf{x})) \mid \mathbf{x} \in X\} \subset \mathbf{R}^n \times \mathbf{R}^m$

Domain of $f = X$, $\qquad$ Codomain of $f = Y$

Range of f = Image of $f = f(X)$
$= \{y \in Y \mid$ there exists $x \in X$ such that $f(x) = y\}$.

$f(x) = x^2$
$f: \mathbb{R} \to \mathbb{R}$
$f: \mathbb{R} \to [0,$
image $= [0,$

---

f is a function if for all x in domain of f, $f(x)$ has a unique value.
I.e, for all $x, y \in X$, if $x = y$, then $f(x) = f(y)$
and for all $x \in X$, $f(x)$ is defined.

NOT FUNCTIONS

$f(x) = \pm\sqrt{x+}$

$g: \mathbb{R} \to \mathbb{R}$
$g(x) = \sqrt{x}$

---

f is 1:1 if $f(x) = f(y)$ implies $x = y$.

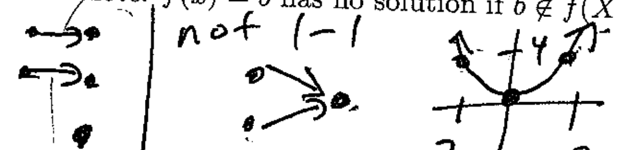
$g(-4)$ not defin
so g not a fu

f gives a one-to-one correspondence between X and $f(X)$.

Given $b \in Y$, $f(x) = b$ has at most one solution

Side-note: $f(x) = b$ has exactly one solution if $b \in f(X)$.
Side-note: $f(x) = b$ has no solution if $b \notin f(X)$.

1-1 $\qquad$ not 1-1



$f(2) = f($
$f(x) = x$
not 1-

---

f is onto if $f(X) = Y$ (i.e., image of f = codomain of f).

Given $b \in Y$, $f(x) = b$ has at least one solution.

$f(x) = -4$
has no sol'n

$f(x) = x^2$
$f: \mathbb{R} \to \mathbb{R}$
not onto

Ex 1: $f: \mathbf{R}^n \to \mathbf{R}$, $f(\mathbf{x}) = ||\mathbf{x}|| = \sqrt{x_1^2 + x_2^2 + ... + x_n^2}$

Domain = $\mathbb{R}^n$ Codomain = $\mathbb{R}$ Image = $[0, \infty)$

Is f 1:1? NO

Proof: $f(1, 0, 0, 0, ..., 0) = \sqrt{1+0+0...0} = 1$
$f(-1, 0, ... 0) = \sqrt{1+0...+0} = 1$

Is f onto? NO

Proof: Codomain $= \mathbb{R}$
Image $= [0, \infty)$ $\mathbb{R} \neq [0, \infty)$

Alternate Proof:
$f(\vec{x}) = -1$ has no sol'n
or -1 is not in the image of f

Ex 2: $g(x, y) = (x^2y, x^4 - y, x^6)$

Domain = $\mathbb{R}^2$ Codomain = $\mathbb{R}^3$

Is g 1:1? NO

Proof: $g(4, 0) = g(-4, 0)$

~~Alt Proof: $g(0, 1) = g(0, -1)$~~

Is g onto?

~~$(0, -1, 0)$~~

Proof:

$g(x, y) = (0, 0, -1)$ has no sol'n
$(0, 0, -1)$ is not in image of g

Ex 3: $h(\mathbf{x}) = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + 2y + 3z \\ 4x + 5y + 6z \end{pmatrix}$

I.e, $h(\mathbf{x}) = (x + 2y + 3z, 4x + 5y + 6z)$.

Domain = $\mathbb{R}^3$ Codomain = $\mathbb{R}^2$ Image =

Is h onto? Is h 1:1?

How many solutions does $h(\mathbf{x}) = \mathbf{b}$ have?

I.e., how many solutions does $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ have?

I.e, how many solutions does the following system of equations have:

$$x + 2y + 3z = b_1,$$

$$4x + 5y + 6z = b_2.$$

Does $\begin{pmatrix} 1 \\ 4 \end{pmatrix} x + \begin{pmatrix} 2 \\ 5 \end{pmatrix} y + \begin{pmatrix} 3 \\ 6 \end{pmatrix} z$ span all of $\mathbf{R}^2$?

Is $\{\begin{pmatrix} 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \end{pmatrix}, \begin{pmatrix} 3 \\ 6 \end{pmatrix}\}$ linearly independent?

$f: X \to \mathbb{R}^2$

$f(\vec{x}) = (f_1(\vec{x}), f_2(\vec{x}))$

$f_1 : X \to \mathbb{R}$, $f_2 : X \to \mathbb{R}$

Definitions:

If the codomain of f is $\mathbf{R}$ (i.e., $f : X \to \mathbf{R}$), we say that f is *real-valued* or *scalar valued.*

Suppose $f : X \subset \mathbf{R}^2 \to \mathbf{R}$ and c is a constant scalar.

The *level curve at height c of f* is the curve in $\mathbf{R}^2$ defined by $f(x,y) = c$. That is,

the level curve at height c of $f = \{(x,y) \in \mathbf{R}^2 \mid f(x,y) = c\}$.

The *contour curve at height c of f* is the curve in $\mathbf{R}^3$ defined by the two equations, $z = f(x,y), z = c$. That is,

the contour curve at height c of $f =$

$$= \{(x,y,z) \in \mathbf{R}^2 \mid z = f(x,y) = c\}$$
$$= \{(x,y,f(x,y)) \in \mathbf{R}^3 \mid f(x,y) = c\}.$$

Recall the graph of $f = \{(x,y,z) \mid z = f(x,y)\}$

$$= \{(x,y,f(x,y)) \mid (x,y) \in X\} \subset \mathbf{R}^2 \times \mathbf{R}$$

The *section of the graph of f by the plane* $x = c$ is the set of points in $\mathbf{R}^3$ defined by the two equations, $z = f(x,y), x = c$. That is,

the section by $x = c$ is $\{(x,y,z) \in \mathbf{R}^2 \mid z = f(x,y), x = c\}$

$$= \{(c,y,f(c,y)) \in \mathbf{R}^3 \mid (c,y) \in X\}.$$

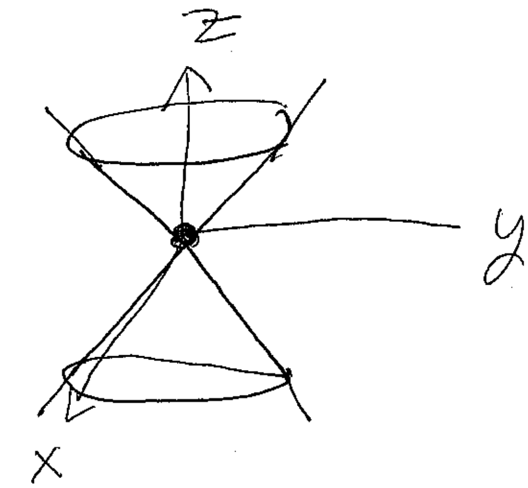
The section by $y = c$ is $\{(x,y,z) \in \mathbf{R}^2 \mid z = f(x,y), y = c\}$

$$= \{(x,c,f(x,c)) \in \mathbf{R}^3 \mid (x,c) \in X\}.$$

$f : \mathbb{R}^2 \to \mathbb{R}$

$f(x,y) = \|(x,y)\|$

$= \sqrt{x^2+y^2}$



Domain $= \mathbb{R}^2 = \{x, y \mid x, y \in$

x	y	$z = f(x,y) = \sqrt{x^2+y^2}$
0	0	0
1	2	$\sqrt{5}$

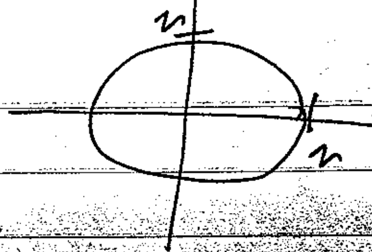
$f : \mathbb{R}^3 \to \mathbb{R}$ graph of $f \subset \mathbb{R}^4$

cs in 2D: $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$

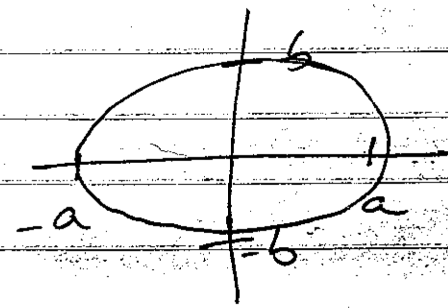
1

1/22/0

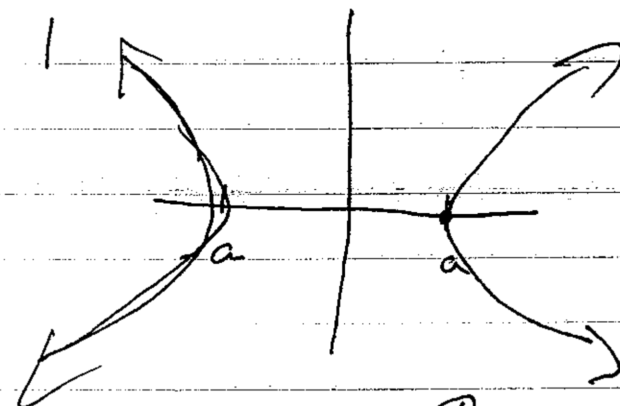
circle: $x^2 + y^2 = r^2$



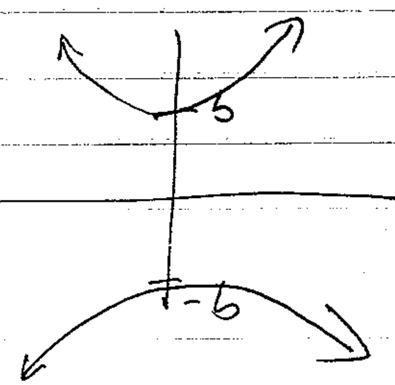
ellipse: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$



hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

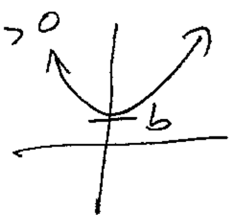


$\dfrac{y^2}{b^2} - \dfrac{x^2}{a^2} = 1$

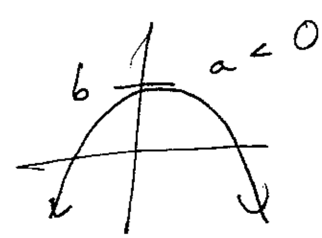


parabola: $y = ax^2 + b$

$a>0$

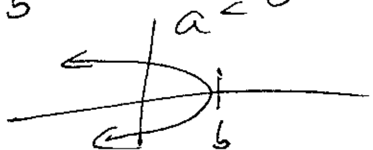


$a<0$

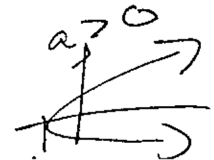


$x = ay^2 + b$

$a<0$



$a>0$



[Conics in $\mathbb{R}^2$: $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$]

for suitable constants $A, \ldots, F$.

In $\mathbf{R}^3$, the analytic analogue of the conic section is called a **quadric surface**. Quadric surfaces are those defined by equations that are polynomials of degree two in three variables:

$$Ax^2 + Bxy + Cxz + Dy^2 + Eyz + Fz^2 + Gx + Hy + Iz + J = 0.$$

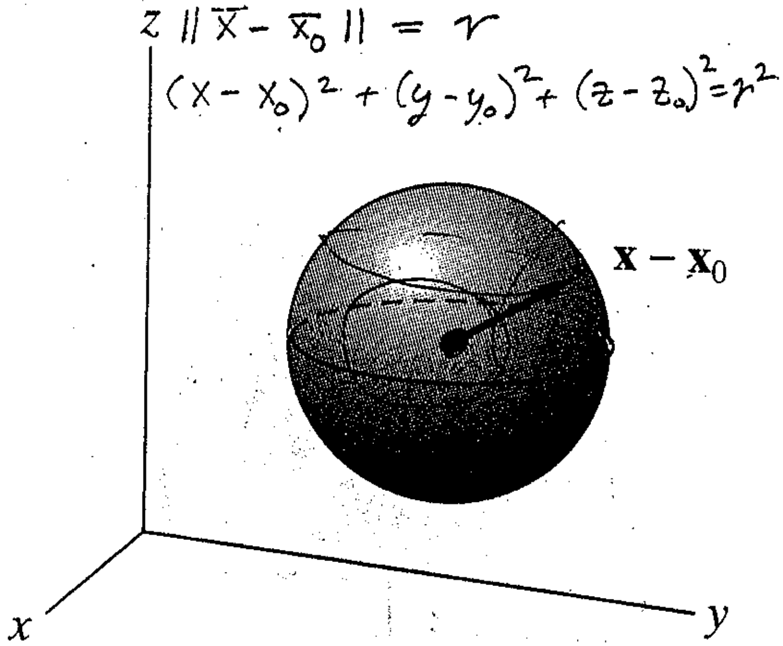


Figure 2.20 The sphere of radius a, centered at (x_0, y_0, z_0).

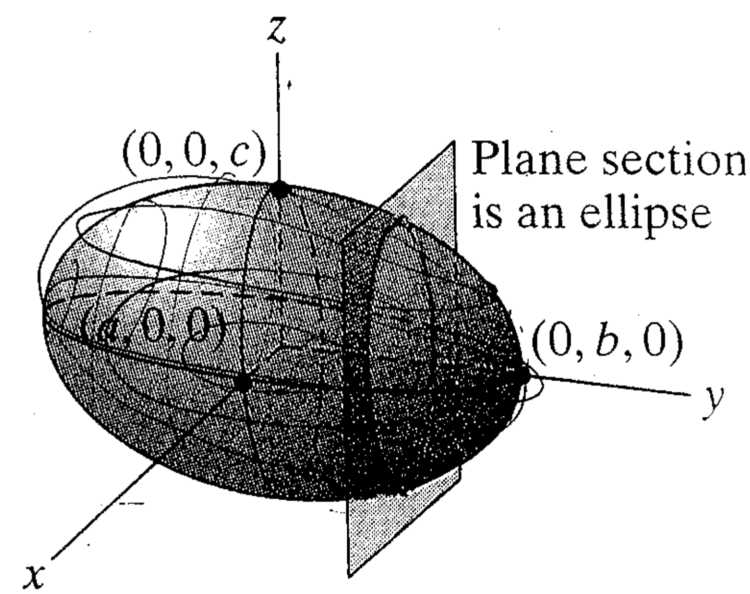


Figure 2.21 The ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1.$

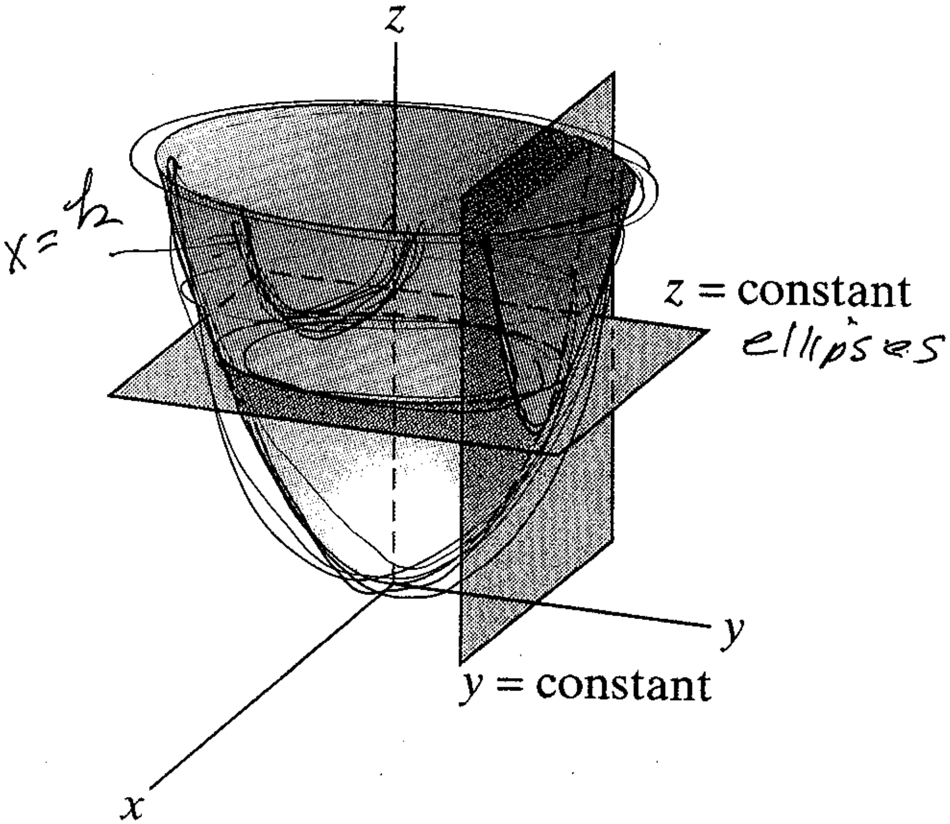


Figure 2.22 The elliptic paraboloid
$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$.

K

$z = K + \frac{y^2}{\sim}$

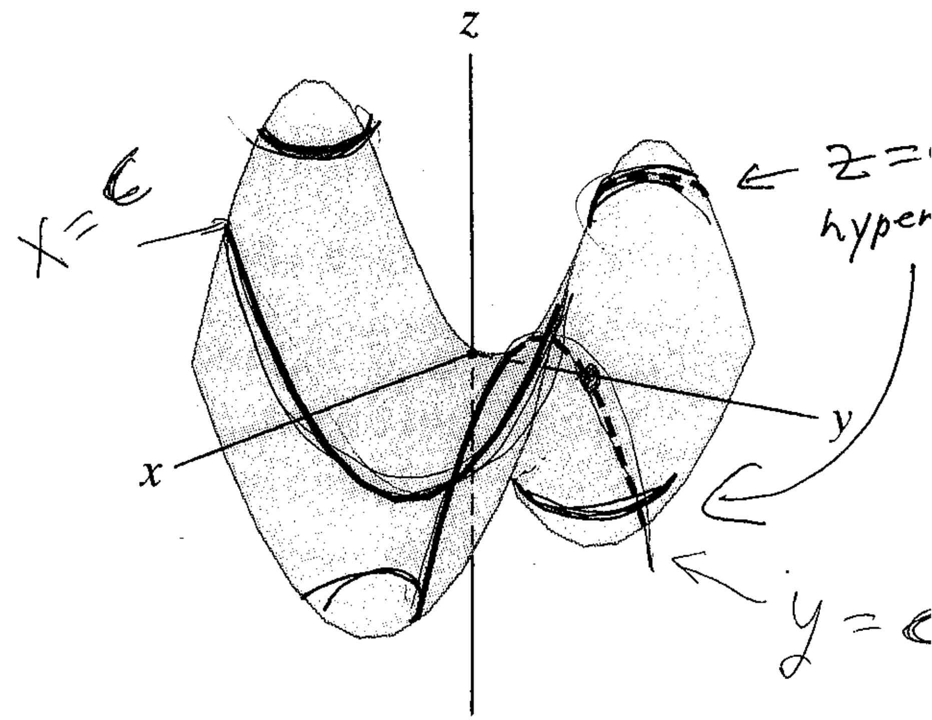


Figure 2.23 The hyperbolic paraboloid $\frac{z}{c} = \frac{y^2}{b^2} - \frac{x^2}{a^2}$.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z^2}{c^2}$$

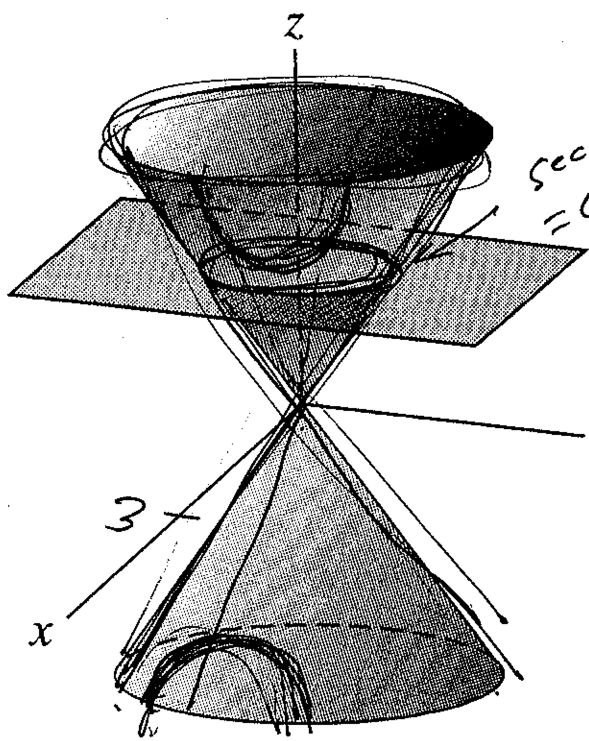


Figure 2.24 The elliptic cone $\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$.

section at
$x = 3$
hyperbola : $\frac{z^2}{c^2} - \frac{y^2}{b^2} = \frac{9}{a^2}$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z^2}{c^2}+1$$

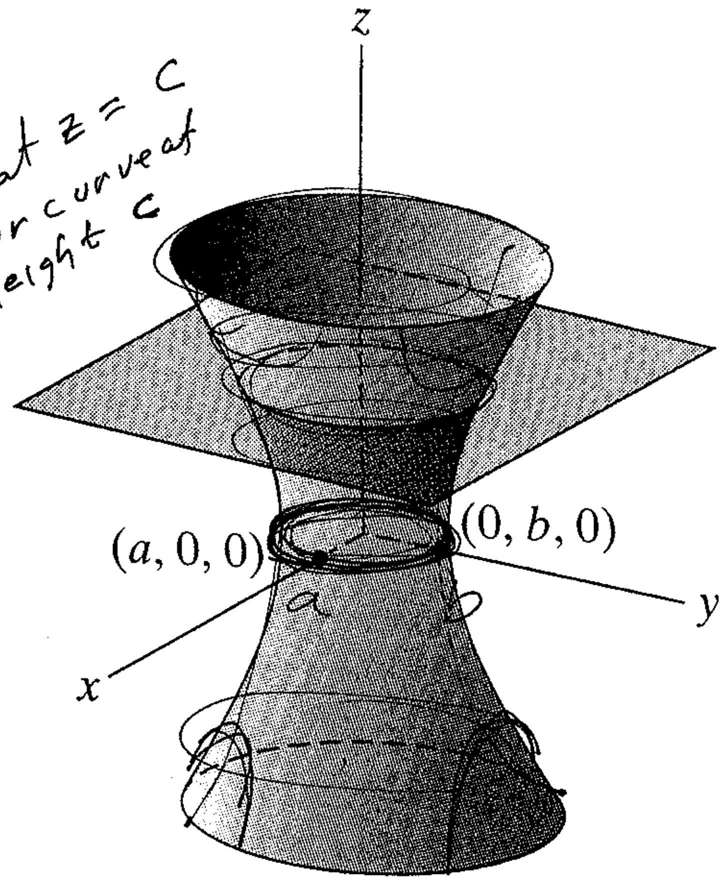


Figure 2.25 The graph of the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ is a hyperboloid of one sheet.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1+\frac{z^2}{c^2}\geq 1$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z^2}{c^2}-$$

$z \neq 0$ need

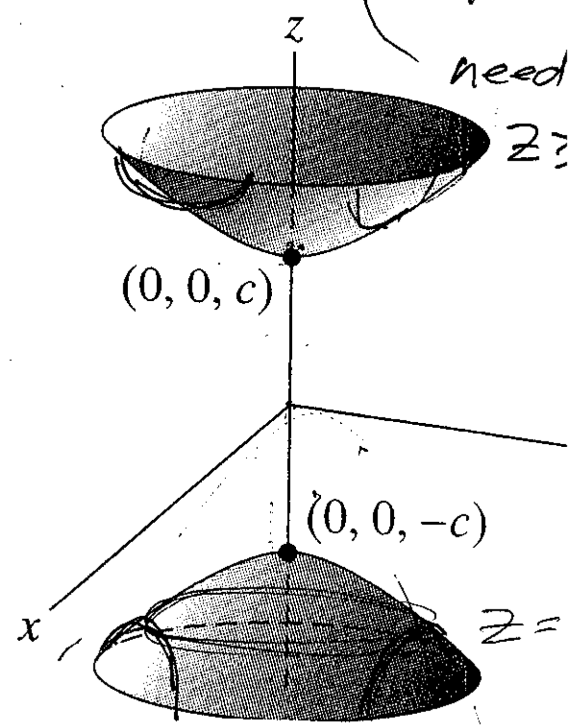


Figure 2.26 The graph of the equation $\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is a hyperboloid of two sheets.

larger $|z|$ $\Rightarrow$ larger ellips