Let $f : X \to Y$ where $X \subset \mathbf{R}^n$, $Y \subset \mathbf{R}^m$

Graph of $f = \{(\mathbf{x}, f(\mathbf{x})) \mid \mathbf{x} \in X\} \subset \mathbf{R}^n \times \mathbf{R}^m$

Domain of $f = X$,   Codomain of $f = Y$,   Image of $f = f(X)$.

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f is a function if for all x in domain of f, $f(x)$ has a unique value.

I.e, for all $x, y \in X$, if $x = y$, then $f(x) = f(y)$

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f is 1:1 if $f(x) = f(y)$ implies $x = y$.

f gives a one-to-one correspondence between X and $f(X)$.

Given $b \in Y$, $f(x) = b$ has at most one solution

Side-note: $f(x) = b$ has exactly one solution if $b \in f(X)$.

. Side-note: $f(x) = b$ has no solution if $b \notin f(X)$.

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f is onto if $f(X) = Y$ (i.e., image of f = codomain of f).

Given $b \in Y$, $f(x) = b$ has at least one solution.

Ex 1: $f : \mathbf{R}^n \to \mathbf{R}$, $f(\mathbf{x}) = ||\mathbf{x}||$

Domain $= \mathbf{R}^n$ Codomain $= \mathbf{R}$ Image $= [0, \infty)$

f is not 1:1: $f(1, 0, ..., 0) = 1 = f(0, 1, 0, ..., 0)$

f is not onto: $f(\mathbf{x}) = -1$ has no solution.

or codomain of $f = \mathbf{R} \neq [0, \infty) =$ image of f

Ex 2: $g(x, y) = (x^2y, x^4 - y, x^6)$

Domain $= \mathbf{R}^2$ Codomain $= \mathbf{R}^3$

g is not 1:1: $g(1, 0) = (0, 1, 1) = g(-1, 0)$

g is not onto: $g(x, y) = (0, 1, -1)$ has no solution.

Ex 3: $h(x) = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

I.e, $h(x) = (x + 2y + 3z, 4x + 5y + 6z)$.

Domain $=$ Codomain $=$ Image $=$

Is h onto? Is h 1:1?

How many solutions does $h(\mathbf{x}) = \mathbf{b}$ have?

I.e., how many solutions does $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ have?

I.e, how many solutions does the following system of equations have:

$$x + 2y + 3z = b_1,$$
$$4x + 5y + 6z = b_2.$$

I.e., does $\begin{pmatrix} 1 \\ 4 \end{pmatrix} x + \begin{pmatrix} 2 \\ 5 \end{pmatrix} y + \begin{pmatrix} 3 \\ 6 \end{pmatrix} z$ span all of $\mathbf{R}^2$?