Theorem [JN, theorem 1.5]

Two generalized Seifert fibrations where $\alpha_i > 0$ for all i are equivalent (up to fiber preserving homeomorphism) if,

1.) They have the same genus.

2.) They can be obtained from each other by the following operations:

i.) Add or delete any Seifert pair $(\alpha, \beta) = (1, 0)$.

ii.) Replace any $(0, \pm 1)$ by $(0, \mp 1)$.

iii.) Replace each (α_i, β_i) by $((\alpha_i, \beta_i + K_i\alpha_i)$ provided $\Sigma K_i = 0$.

A Seifert fibration has a unique normal form

$M(g; (1, \beta_0), (\alpha_1, \beta_1), ..., (\alpha_n, \beta_n))$ where $0 < \beta_i < \alpha_i$.

A generalized Seifert fibration can be uniquely represented as

$M(g; (0, 1), ..., (0, 1), (\alpha_1, \beta_1), ..., (\alpha_n, \beta_n))$ where $0 < \beta_i < \alpha_i$.

Defn: The euler number of a generalized Seifert fibration is $e(M \to F) = -\Sigma \frac{\alpha_i}{\beta_i}$

The euler number $\neq \infty$ iff $M \to F$ is a true Seifert fibration.

$M(g; (\alpha_1, \beta_1), ..., (\alpha_n, \beta_n)) = -M(g; (\alpha_1, -\beta_1), ..., (\alpha_n, -\beta_n))$

Theorem [JN, theorem 6.1] The following following determine the classification of closed Seifert fiberable manifolds up to (not necessarily fiber preserving) orientation preserving homeomorphism:

1) $M(-1;(\alpha,\beta)) = M(0,(2,1),(2,-1),(-\beta,\alpha))$

$M(-2;(1,0)) = M(0,(2,1),(2,1),(2,-1),(2,-1))$

2.) The diffeomorphism in (1) and the Seifert fibered structures on lens spaces are the only examples of 3-manifolds have non isomorphic (true) Seifert fibrations.

3.) $M(g;(0,1),(\alpha_1,\beta_1),...,(\alpha_n,\beta_n))$
$$= \begin{cases} [\#_{i=1}^{2g}(S^1\times S^2)]\#[\#_{i=1}^{n}L(\alpha_i,-\beta_i)] & \text{if } g\geq 0 \\ [\#_{i=1}^{|g|}(S^1\times S^2)]\#[\#_{i=1}^{n}L(\alpha_i,-\beta_i)] & \text{if } g<0 \end{cases}$$

The only (true) Seifert fibered manifold which is not prime is $M(-1;(1,0) = RP^3\#RP^3$.

Theorem [JN, theorem 5.2] (Waldhausen)

Let M_1, M_2 Seifert fibered and not in the following list:

1.) lens spaces

2.) $M(0;(\alpha_1,\beta_1),(\alpha_2,\beta_2),(\alpha_3,\beta_3))$

3.) $M(1;(1,0)) = T^3$

4.) As is part one of theorem 5.1

Then any homeomorphis $M_1 \to M_2$ is isotopic to a fiber preserving homeomorphism.

Theorem [JN, theorem 6.1] Let $M = M(g;(\alpha_1,\beta_1),...,(\alpha_n,\beta_n))$.

1.) If $g \geq 0$, i.e., the orbifold is orientable:
$\pi_1(M) =< a_i, b_i, q_j, f \mid [f,a_i] = [f,b_i] = [f,q_j] = 1, q_j^{\alpha_j} h^{\beta_j} = 1,$
$q_1 q_2 \cdots q_n [a_1,b_1] \cdots [a_g,b_g] = 1, i = 1,...,g, j = 1,...,n >$.

2.) If $g < 0$, i.e., the orbifold is non-orientable:

$$\pi_1(M) =< a_i, q_j, f \mid a_i^{-1} f a_i = f^{-1}, [f,q_j] = 1, q_j^{\alpha_j} f^{\beta_j} = 1,$$
$$q_1 q_2 \cdots q_n a_1^2 \cdots a_{|g|}^2 = 1, i = 1,...,|g|, j = 1,...,n > .$$

Orientable Manifolds:

Oo: All curves have value +1; punctured surface $\times S^1$.

On: All one-sided curves have value -1.

Non-orientable Manifolds:

No: There are curves of value -1.

NnI: All curves have value +1; punctured surface $\times S^1$.

NnII: There are one-sided curves of value -1 and of value +1; each orientation producing simple closed curve has value -1.

NnIII: There are one-sided curves of value -1 and of value +1; each orientation producing simple closed curve has value +1.