Biology application: Suppose the number of bacteria grow at an average rate or r = 10% per hour. If the initial population is 100 bacteria, find the number of bacteria after 1 hour, after 2 hours, after T hours.

Identical application, but in Finance:

Let $P(t)$ = amount in an account at time t (in years).

Ex 1: Suppose \$100 is deposited in the account earning an interest rate of $r = 10\%$ per year. Find the amount in the account after 1 year, after 2 years, after 3 years, and after T years.

$t = 0$: $P(0) = \$100$

$t = 1$: $P(1) = 100(1 + 0.1) = 100(1.1) = \$110$

$t = 2$: $P(2) = 100(1+0.1)(1+0.1) = 100(1+0.1)^2 = 100(1.1)^2 = \$121$

$t = 3$: $P(3) = 100(1 + 0.1)^3 = \$100(1.1)^3 = 133.10$

$\vdots$

$t = T$: $P(T) = 100(1 + 0.1)^T = \$100(1.1)^T$

The average interest rate earned is 10% per year.

The average rate of change in the account btwn year 0 and year 1:

$$\frac{P(1)-P(0)}{1} = 100(1.1) - 100 = 100(0.1) = \$10/\text{year}.$$

The average rate of change between year t and year $t+1$:

$$\frac{P(t+1)-P(t)}{1} = 100(1.1)^{t+1} - 100(1.1)^t = \$100(1.1)^t(0.1)/\text{year}.$$

Instantaneous rate of change at time t:

$$P'(t) = [100(1.1)^t]' = 100ln(1.1)(1.1)^t = (9.53102...) \cdot (1.1)^t$$

At $t = 1$: $P'(1) = 100ln(1.1)(1.1) = 10.48...$

Ex 2: Suppose \$100 is deposited in the account earning an interest rate of $r = 10\%$ per year, compounded monthly. Find the amount in the account after 1 month, 1 year, 2 years, and T years.

$t = 0$: $P(0) = \$100$

$t = 1$ month: $P(\frac{1}{12}) = 100(1 + \frac{0.1}{12}) = \$100.83$

$t = 1$ year: $P(1) = 100(1 + \frac{0.1}{12})^{12} = \$110.47$

$t = 2$ years: $P(2) = 100(1 + \frac{0.1}{12})^{12\cdot 2} = \$122.04$

$\vdots$

$t = T$ years: $P(T) = 100(1 + \frac{0.1}{12})^{12\cdot T} = \$100(1.1047...)^T$

The average interest rate earned is $\frac{10}{12}\%$ per month.

The average interest rate earned is 10.47...% per year.

The average rate of change between year t and year $t + 1$:

$$\begin{aligned}\frac{P(t+1)-P(t)}{1} &= 100(1 + \tfrac{0.1}{12})^{12(t+1)} - 100(1 + \tfrac{0.1}{12})^{12t} \\ &= \$100(1 + \tfrac{0.1}{12})^{12t}[(1 + \tfrac{0.1}{12})^{12} - 1]/\text{year}.\end{aligned}$$

The approximate average rate of change between year t and year $t+1$:

$$\begin{aligned}\frac{P(t+1)-P(t)}{1} &= 100(1.1047)^{t+1} - 100(1.1047)^{t} \\ &= \$100(1.1047)^{t}(0.1047)/\text{year}.\end{aligned}$$

The instantaneous rate of change at time t:

$$\begin{aligned}P'(t) = [100(1 + \tfrac{0.1}{12})^{12\cdot t}]' &= 100ln[(1 + \tfrac{0.1}{12})^{12}] \cdot [(1 + \tfrac{0.1}{12})^{12}]^{t} \\ &= 1200ln(1 + \tfrac{0.1}{12}) \cdot (1 + \tfrac{0.1}{12})^{12t} \\ &= (9.95856...) \cdot (1 + \tfrac{0.1}{12})^{12t}\end{aligned}$$

At $t = 1 : P'(1) = 1200ln(1 + \frac{0.1}{12}) \cdot (1 + \frac{0.1}{12})^{12} = 11.001...$

At $t = \frac{1}{12}$, $P'(\frac{1}{12}) = 1200ln(1 + \frac{0.1}{12}) \cdot (1 + \frac{0.1}{12}) = 10.0416$

Ex 3: Suppose \$100 is deposited in the account earning an interest rate of $r = 10\%$ per year, compounded daily. Find the amount in the account after 1 day, 1 year, 2 years, and T years.

$t = 0$: $P(0) = \$100$

$t = 1$ day: $P(\frac{1}{356}) = 100(1 + \frac{0.1}{365}) = \$100.03$

$t = 1$ year: $P(1) = 100(1 + \frac{0.1}{365})^{365} = \$110.52$

$t = 2$ years: $P(2) = 100(1 + \frac{0.1}{365})^{365 \cdot 2} = \$122.14$

$\vdots$

$t = T$ years: $P(T) = 100(1 + \frac{0.1}{365})^{365 \cdot T} = \$100(1.10515578...)^T$

The average interest rate earned is $\frac{10}{365}\%$ per day.

The average interest rate earned is 10.515578...% per year.

The average rate of change between year t and year $t + 1$:

$$\frac{P(t+1) - P(t)}{1} = 100(1 + \frac{0.1}{365})^{365(t+1)} - 100(1 + \frac{0.1}{365})^{365t}$$
$$= \$100(1 + \frac{0.1}{365})^{365t}[(1 + \frac{0.1}{365})^{365} - 1]/\text{year}.$$

The instantaneous rate of change at time t:

$$P'(t) = [100(1 + \frac{0.1}{365})^{365 \cdot t}]' = 100ln[(1 + \frac{0.1}{365})^{365}] \cdot [(1 + \frac{0.1}{365})^{365}]^t$$
$$= 36500ln(1 + \frac{0.1}{365}) \cdot (1 + \frac{0.1}{365})^{365t}$$
$$= (9.99863...) \cdot (1 + \frac{0.1}{365})^{365t}$$

At $t = 1 : P'(1) = 36500ln(1 + \frac{0.1}{365}) \cdot (1 + \frac{0.1}{365})^{365} = 11.05...$

At $t = \frac{1}{365}$, $P'(\frac{1}{365}) = 36500ln(1 + \frac{0.1}{365}) \cdot (1 + \frac{0.1}{365}) = 10.00...$

Ex 4: Suppose \$100 is deposited in the account earning an interest rate of $r = 10\%$ per year, compounded n times per year. Find the amount in the account after T years.

$t = T$ years: $P(T) = 100(1 + \frac{0.1}{n})^{n \cdot T}$

Ex 5: Suppose \$100 is deposited in the account earning an interest rate of $r = 10\%$ per year, compounded continuously. Find the amount in the account after T years.

$t = T$ years: $P(T) = lim_{n \to \infty} 100(1 + \frac{0.1}{n})^{n \cdot T} = 100e^{0.1T}$

Definition $e = lim_{n \to \infty}(1 + \frac{1}{n})^n = 2.7...$

FYI : By Taylor series approximation from Calculus II

$$e^{0.1T} = lim_{n \to \infty}(1 + \frac{1}{n})^{n(0.1)T} = lim_{n \to \infty}[(1 + \frac{1}{n})^{0.1}]^{nT}$$

$$= lim_{n \to \infty}[1 + \frac{0.1}{n} - \frac{0.045}{n^2} + \frac{0.0285}{n^3} - ...]^{nT} = lim_{n \to \infty}[1 + \frac{0.1}{n}]^{nT}$$

Ex 6: Suppose $\$P_0$ is deposited in the account earning an interest rate of $r = s\%$ per year ($r = \frac{s}{100}$), compounded continuously.
t years: $P(t) = lim_{n \to \infty} P_0(1 + \frac{r}{n})^{n \cdot t} = P_0 e^{rt}$

Ex 7: Suppose \$1 is deposited in the account earning an interest rate of $r = 10\%$ per year ($r = \frac{100}{100} = 1$), compounded continuously.
t years: $P(t) = lim_{n \to \infty}(1 + \frac{.1}{n})^{n \cdot t} = e^{0.1t}$

Note the instantaneous rate of change is $10\% = 0.1e^{0.1t}$

That is $P'(t) = [e^{0.1t}]' = 0.1e^{0.1t}$

Ex 8: Suppose \$1 is deposited in the account earning an interest rate of $r = 100\%$ per year ($r = \frac{100}{100} = 1$), compounded continuously.
t years: $P(t) = lim_{n \to \infty}(1 + \frac{1}{n})^{n \cdot t} = e^t$

Note the instantaneous rate of change is $100\% = e^t$

That is $P'(t) = [e^t]' = e^t$