CH 8: Solve $\frac{dy}{dt} = f(t, y)$

8.1: Direction Fields **************************

Ch 5: Solve $\frac{dy}{dt} = f(t)$

Suppose f is continuous

$$dy = f(t)dt$$

$$\int dy = \int f(t)dt$$

$y = F(t) + C$ where F is any anti-derivative of F.

Initial Value Problem (IVP): $y(t_0) = y_0$

$$y_0 = F(t_0) + C \text{ implies } C = y_0 - F(t_0)$$

Hence unique solution (if domain connected) to IVP:

$$y = F(t) + y_0 - F(t_0)$$

Section 8.3: Solve $\frac{dy}{dt} = f(y)$

If given either differential equation $y' = f(y)$ OR direction field:

Find equilibrium solutions and determine if stable, unstable, semi-stable.

Understand what the above means.

8.2 Linear vs Non-linear

nonlinear: $y' = y^2$

linear: $a_0(t)y^{(n)} + ... + a_n(t)y = g(t)$

Ex: $ty'' - t^3y' - 3y = sin(t)$

Ex: $2y'' - 3y' - 3y^2 = 0$

First order linear eqn: $y' + p(t)y = g(t)$

Ex 1: $y' = ay + b$

Ex 2: $y' + 3t^2y = t^2$, $y(0) = 0$

Note: could use section 8.4 method, separation of variables to solve ex 1 and 2.

Ex 3: $t^2y' + 2ty = sin(t)$

FYI: Solve $t^2y' + 2ty = sin(t)$
(note, cannot use separation of variables).

$t^2y' + 2ty = sin(t)$

$(t^2y)' = sin(t)$

$\int (t^2y)'dt = \int sin(t)dt$

$(t^2y) = -cos(t) + C$ implies $y = -t^{-2}cos(t) + Ct^{-2}$

FYI: Solve $y' + p(x)y = g(x)$

Let $F(x)$ be an anti-derivative of $p(x)$

$e^{F(x)}y' + [p(x)e^{F(x)}]y = g(x)e^{F(x)}$

$e^{F(x)}y' + [F'(x)e^{F(x)}]y = g(x)e^{F(x)}$

$[e^{F(x)}y]' = g(x)e^{F(x)}$

$e^{F(x)}y = \int g(x)e^{F(x)}dx$

$y = e^{-F(x)} \int g(x)e^{F(x)}dx$

8.4: Solve $\frac{dy}{dt} = ay + b$ by separating variables: $\frac{dy}{ay+b} = dt$

$\int \frac{dy}{ay+b} = \int dt$

$\frac{ln|ay+b|}{a} = t + C$

$ln|ay + b| = at + C$

$e^{ln|ay+b|} = e^{at+C}$

$|ay + b| = e^C e^{at}$

$ay + b = \pm(e^C e^{at})$

$ay = Ce^{at} - b$ implies $y = Ce^{at} - \frac{b}{a}$

Ex 2: $y' + 3t^2 y = t^2$, $y(0) = 0$

$\frac{dy}{dt} + 3t^2 y = t^2$

$\frac{dy}{dt} = t^2 - 3t^2 y$

$dy = (t^2 - 3t^2 y)dt$

$dy = (1 - 3y)t^2 dt$

$\int \frac{dy}{1-3y} = \int t^2 dt$

$-\frac{1}{3} ln|1 - 3y| = \frac{t^3}{3} + C$

$ln|1 - 3y| = -t^3 + C$

$|1 - 3y| = e^{-t^3+C}$

$|1 - 3y| = e^C e^{-t^3}$

$1 - 3y = \pm e^C e^{-t^3}$

$1 - 3y = Ce^{-t^3}$

$-3y = Ce^{-t^3} - 1$

$y = \frac{1-Ce^{-t^3}}{3} = \frac{1}{3} - Ce^{-t^3}$

$e^{-\frac{1}{3} ln|1-3y|} = e^{-\frac{t^3}{3}+C}$

$(e^{ln|1-3y|})^{-\frac{1}{3}} = e^{-\frac{t^3}{3}+C}$

$ln(|1 - 3y|^{-\frac{1}{3}}) = \frac{t^3}{3} + C$

$|1 - 3y|^{-\frac{1}{3}} = e^{-\frac{t^3}{3}+C}$

$|1 - 3y|^{-\frac{1}{3}} = e^C e^{\frac{t^3}{3}}$

$(|1 - 3y|^{-\frac{1}{3}})^{-3} = (e^C e^{\frac{t^3}{3}})^{-3}$

$|1 - 3y| = (e^{-3C} e^{-t^3})$

$1 - 3y = Ce^{-t^3}$

**************Existence of solution**************

Ch 5) $y' = f(t)$, solution exists if f is continuous
$y = F(t) + C$ where F is any anti-derivative of f.

8.2) linear: $y' + p(x)y = g(x)$, solution exists if p and g are continuous

Ch 8): $y' = f(t, y)$, solution may or may not exist.

Ex: $y' = y' + 1$

Ex: $(y')^2 = -1$

IVP ex: $\frac{dy}{dx} = y(1 + \frac{1}{x})$, $y(0) = 1$

$\int \frac{dy}{y} = \int (1 + \frac{1}{x}) dx$

$ln|y| = x + ln|x| + C$

$|y| = e^{x+ln|x|+C} = e^x e^{ln|x|} e^C = C|x|e^x = Cxe^x$

$y = \pm Cxe^x$ implies $y = Cxe^x$

$y(0) = 1$: $\quad 1 = C(0)e^0 = 0$ implies

$$\text{IVP } \frac{dy}{dx} = y(1 + \frac{1}{x}),\ y(0) = 1 \text{ has no solution.}$$

See direction field created using
www.math.rutgers.edu/~ sontag/JODE/JOdeApplet.html

**************Uniqueness of solution**************

Given an initial value problem,

Ch 5) $y' = f(t)$, $y(t_0) = y_0$: if f continuous, then on appropriate domain, unique solution $y = F(t) + y_0 - F(t_0)$.

8.2) linear: $y' + p(x)y = g(x)$, then on appropriate domain, unique solution if p and g are continuous .

Ch 8): $y' = f(t, y)$, solution may or may not be unique.

Ex: $y' = y^{\frac{1}{3}}$

Note $y = 0$ is a solution to $y' = y^{\frac{1}{3}}$ since $y' = 0 = 0^{\frac{1}{3}} = y^{\frac{1}{3}}$

Suppose $y \neq 0$. Then $\frac{dy}{dx} = y^{\frac{1}{3}}$ implies $y^{-\frac{1}{3}} dy = dx$

$\int y^{-\frac{1}{3}} dy = \int dx$ implies $\frac{3}{2} y^{\frac{2}{3}} = x + C$

$y^{\frac{2}{3}} = \frac{2}{3}x + C$ implies $y = \pm\sqrt{(\frac{2}{3}x + C)^3}$

Suppose $y(3) = 0$. Then $0 = \sqrt{(2 + C)^3}$ implies $C = -2$.

Thus initial value problem, $y' = y^{\frac{1}{3}}$, $y(3) = 0$, has 3 sol'ns:

$$y = 0, \quad y = \sqrt{(\tfrac{2}{3}x - 2)^3}, \quad y = -\sqrt{(\tfrac{2}{3}x - 2)^3}$$