Let $N(d_1, ..., d_n) =$ the number of labeled trees with n vertices $\{v_1, ..., v_n\}$ such that $deg(v_i) = d_i + 1$.

Let $C(n-2, d_1, ..., d_n) = \frac{(n-2)!}{d_1!d_2!\cdots d_n!}$.

section 3.5 **34e.)** Claim: $N(d_1, ..., d_n) = \begin{cases} C(n-2; d_1, ..., d_n) & \text{if } \Sigma d_i = n-2 \\ 0 & \text{otherwise} \end{cases}$

Claim $N(d_1, ..., d_n) = 0$ if $\Sigma d_i \neq n-2$

Proof:
$\Sigma d_i = \Sigma_{i=1}^{n}(deg(v_i) - 1) = [\Sigma_{i=1}^{n}(deg(v_i)] - n = [2(n-1)] - n = 2n - 2 - n = n - 2$

Thus $\Sigma d_i = n - 2$. Hence $N(d_1, ..., d_n) = 0$ if $\Sigma d_i \neq n - 2$.

Claim $N(d_1, ..., d_n) = C(n-2; d_1, ..., d_n)$ if $\Sigma d_i = n - 2$ (*)

Proof by induction on $k =$ number of vertices.

By part a, the equality holds for $n = 2$.

Induction hypothesis: Suppose (*) is true when $k = n - 1$.

By part b, $d_j = 0$ for some j. WLOG assume $j = n$. Thus by part c,

$N(d_1, ..., d_n) = N(d_1, ..., d_{n-1}, 0) = N(d_1-1, d_2, ..., d_{n-1}) + N(d_1, d_2-1, d_3..., d_{n-1}) + ... + N(d_1, ..., d_{n-2}, d_{n-1} - 1)$.

By part d,

$C(n-2; d_1, ..., d_n) = C(n-2; d_1, ..., d_{n-1}, 0) = C(n-3; d_1 - 1, d_2, ..., d_{n-1}) + C(n-3; d_1, d_2 - 1, d_3..., d_{n-1}) + ... + C(n-3; d_1, ..., d_{n-2}, d_{n-1} - 1)$.

By the induction hypothesis, $N(d_1, ..., d_n) = N(d_1, ..., d_{n-1}, 0) = N(d_1-1, d_2, ..., d_{n-1}) + N(d_1, d_2-1, d_3..., d_{n-1}) + ... + N(d_1, ..., d_{n-2}, d_{n-1}-1) = C(n-3; d_1-1, d_2, ..., d_{n-1}) + C(n-3; d_1, d_2-1, d_3..., d_{n-1}) + ... + C(n-3; d_1, ..., d_{n-2}, d_{n-1}-1) = C(n-2; d_1, ..., d_{n-1}, 0) = C(n-2; d_1, ..., d_n)$.

34a). Suppose $n = 2$. A tree with 2 vertices has 1 edge. Thus $deg(v_i) = 1$ for $i = 1, 2$. Thus $d_i = 0$ for $i = 1, 2$. There is exactly one labeled tree with 2 vertices, $T = (\{v_1, v_2\}, \{\{v_1, v_2\}\})$.

Thus $N(0,0) = 1$. $C(0;0,0) = \frac{0!}{0!0!} = 1$. Thus (*) holds for $n = 2$.

34b.) Claim $d_i = 0$ for some i.

Suppose $d_i > 0$ for all i. Then $deg(v_i) = d_i + 1 > 1$ for all i. That is, $deg(v_i) \geq 2$ for all i.

The number of edges in a graph $= \frac{1}{2}\Sigma deg(v_i)$.

The number of edges in a tree with n vertices is $n - 1$.

Thus $n - 1 = \frac{1}{2}\Sigma_{i=1}^{n} deg(v_i) \geq \frac{1}{2}\Sigma_{i=1}^{n} 2 = \frac{1}{2}(2n) = n$, a contradiction. Thus $d_i = 0$ for some i.

34c). Let A = set of all labeled trees with n vertices $\{v_1, ..., v_n\}$ such that $deg(v_i) = d_i + 1$.

Then $|A| = N(d_1, ..., d_n)$.

For $j = 1, ..., n-1$, let B_j = set of all labeled trees with $n-1$ vertices $\{v_1, ..., v_{n-1}\}$ such that $deg(v_i) = d_i + 1$, $i \neq j$ and $deg(v_j) = (d_j - 1) + 1$.

Then $|B_j| = N(d_1, ..., d_{j-1}, d_j - 1, d_{j+1}, ..., d_{n-1})$ for $j = 1, ..., n-1$.

Note if $T_j \in B_j$, then $deg(v_j) = d_j$.

Suppose $k \neq j$. If $T_k \in B_k$, then $deg(v_j) = d_j + 1$. Thus T_j is not isomorphic to T_j.

Thus $B_j \cap B_k = \emptyset$ for $k \neq j$.

Claim: There exists a bijection $f : A \to \cup_{i=1}^{n-1} B_i$.

Note if the claim is true, then $|A| = |\cup_{i=1}^{n-1} B_i| = \Sigma_{i=1}^{n-1} |B_i|$, since the B_i are pairwise disjoint.

Define $g : \cup_{i=1}^{n-1} B_i \to A$. Let $T = (V, E) \in B_j$. Let $g(T) = (V \cup \{v_n\}, E \cup \{\{v_j, v_n\}\})$. Note $g(T)$ has n vertices and $deg(v_i) = d_i + 1$ for $i = 1, ..., n$. Thus $g : \cup_{i=1}^{n-1} B_i \to A$ is well-defined.

Claim g^{-1} exists.

WLOG assume $d_n = 0$ (relabel the vertices if needed). $d_n = 0$ implies $deg(v_n) = 1$. Suppose the vertex adjacent to v_n is labeled v_j. Let $T'(V', E') \in A$. Define $f : A \to \cup_{i=1}^{n-1} B_i$ by $f(T') = (V' - \{v_n\}, E' - \{\{v_j, v_n\}\})$. Note that $f(T')$ has $n-1$ vertices, $\{v_1, ..., v_{n-1}\}$ and $deg(v_j) = d_j$, $deg(v_i) = d_i + 1$ for $i \neq j$. Thus $f(T')$ is in B_j, and hence f is well-defined.

$f(g((V, E))) = f((V \cup \{v_n\}, E \cup \{\{v_j, v_n\}\}) = (V, E)$.

$g(f((V, E))) = g((V - \{v_n\}, E - \{\{v_j, v_n\}\}) = (V, E)$.

Thus g is invertible. Thus g is a bijection. Thus $|A| = |\cup_{i=1}^{n-1} B_i| = \Sigma_{i=1}^{n-1} |B_i|$.

Alternate proof that g is a bijection:

Claim: g is onto. Let $T' = (V', E') \in A$. Let $T = (V' - \{v_n\}, E' - \{\{v_j, v_n\}\})$. Then $g(T) = g((V' - \{v_n\}, E' - \{\{v_j, v_n\}\}) = (V', E') = T'$.

Claim g is 1-1:

Suppose $g(T) = g(S)$. Claim T and S are isomorphic ...

34d). Note that by the right-hand side of the equation, we are given that

$\Sigma_{i=1}^{n-1} d_i = [\Sigma_{i=1}^{n} d_i] - d_n = n - 2 - 0 = n - 2$

$\Sigma_{i=1}^{n-1} C(n-3; d_1, ..., d_{i-1}, d_i - 1, d_{i+1}, ..., d_{n-1}) = \Sigma_{i=1}^{n-1} \frac{(n-3)!}{d_1! \cdots d_{i-1}!, (d_i - 1)!, d_{i+1}! \cdots d_{n-1}!}$

$= \Sigma_{i=1}^{n-1} \frac{(n-3)! d_i}{d_1! \cdots d_{i-1}!, d_i!, d_{i+1}! \cdots d_{n-1}!}$

$= \frac{(n-3)!}{d_1! \cdots d_{i-1}!, d_i!, d_{i+1}! \cdots d_{n-1}!} \Sigma_{i=1}^{n-1} d_i$

$= \frac{(n-3)!}{d_1! \cdots d_{n-1}!} (n-2)$

$= \frac{(n-2)!}{d_1! \cdots d_{n-1}! 0!} = \frac{(n-2)!}{d_1! \cdots d_{n-1}! d_n!} = C(n-2, d_1, ..., d_n)$