5.2: $(x+y)^n = \Sigma_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$

P.O.: The terms of $(x+y)^n$ are of the form $x^k y^{n-k}$.

The coefficient of $x^k y^{n-k}$

= the number of ways to choose k x's and $(n-k)$ y's

= the number of ways to choose k x's from n x's = $\binom{n}{k}$.

Alternatively,

The coefficient of $x^k y^{n-k}$

= the number of ways to choose k x's and $(n-k)$ y's

= the number of permutations of the multiset

$$\{k \cdot x, (n-k) \cdot y\} = \binom{n}{k}.$$

Obtain other formulas via substitution and algebraic manipulation such as differentiation.

Let $r \in \mathcal{R}$, $k \in \mathcal{Z}$.

Define $\binom{r}{k} = \begin{cases} \frac{r(r-1)...(r-k+1)}{k!} & \text{if } k \geq 1 \\ 1 & \text{if } k = 0 \\ 0 & \text{if } k \leq -1 \end{cases}$

Thm 5.3.1: Let n be a positive integer. The sequence of binomial coefficients is a unimodal sequence. In particular

if n is even, $\binom{n}{0} < \binom{n}{1} ... < \binom{n}{\frac{n}{2}}$

$$\binom{n}{\frac{n}{2}} > ... > \binom{n}{n-1} > \binom{n}{n}$$

and if n is odd

$$\binom{n}{0} < \binom{n}{1} ... < \binom{n}{\frac{n-1}{2}} = \binom{n}{\frac{n+1}{2}}$$

$$\binom{n}{\frac{n+1}{2}} > ... > \binom{n}{n-1} > \binom{n}{n}$$

Proof idea: Look at $\frac{\binom{n}{k}}{\binom{n}{k-1}} = \frac{n-k+1}{k}$

5.4: Multinomial thm

Define $\binom{n}{n_1 n_2 ... n_t} = \frac{n!}{n_1! n_2! ... n_t!}$

Thm 5.5.1: Let $n \in \mathcal{Z}$. Then

$$(x_1 + x_2 + ... x_t)^n = \Sigma \binom{n}{n_1 n_2 ... n_t} x_1^{n_1} x_2^{n_2} + ... x_t^{n_t}$$

where the summation extends over all nonnegative integral solutions to $n_1 + n_2 + ... + n_t = n$

5.5: Newton's Binomial Theorem

Let $r \in \mathcal{R}$, $k \in \mathcal{Z}$.

Define $\begin{pmatrix} r \\ k \end{pmatrix} = \begin{cases} \frac{r(r-1)\ldots(r-k+1)}{k!} & \text{if } k \geq 1 \\ 1 & \text{if } k = 0 \\ 0 & \text{if } k \leq -1 \end{cases}$

Thm 5.5.1: Let $\alpha \in \mathcal{R}$. Then if $0 \leq |x| < |y|$,

$$(x+y)^{\alpha} = \Sigma_{k=0}^{\infty} \begin{pmatrix} \alpha \\ k \end{pmatrix} x^k y^{\alpha-k}$$