Math 150 Exam 2
October 30, 2009

Choose 6 from the following 8 problems. Circle your choices: 1 2 3 4 5 6 7 8
You may do more than 6 problems in which case one of your two unchosen problems can replace your lowest problem at 4/5 the value as discussed in class.

1.) $\binom{2.3}{4} = \frac{(2.3)(1.3)(0.3)(-0.7)}{(4)(3)(2)(1)}$

2a.) State the axiom of choice (you can give either a formal or informal definition).

Formal: Suppose $\{C_\alpha \mid \alpha \in A\}$ is an infinite collection of sets (i.e, $|A|$ is infinite). Then we can form a set $B = \{x_\alpha \mid \alpha \in A\}$ by taking one element $x_\alpha \in C_\alpha$ for each C_α (i.e., for each $\alpha \in A$).

Alternate formal definition: Given an infinite collection of sets $\{C_\alpha \mid \alpha \in A\}$, we can define a function $f : \{C_\alpha \mid \alpha \in A\} \to \cup_{\alpha \in A} C_\alpha$ such that $f(C_\alpha) \in C_\alpha$.

Informal: If you have an infinite collection of pairs of socks, you can choose one sock from each pair.

2b.) State a cyclic Gray code of order 3.

0 0 0
0 0 1
0 1 1
0 1 0
1 1 0
1 1 1
1 0 1
1 0 0

3.) Let $\mathcal{P} = \{P_\alpha \mid \alpha \in A\}$ be a partition of X. Define a relation $\sim$ on X by $x \sim y$ if and only if there exists $P_\alpha \in \mathcal{P}$ such that $x, y \in P_\alpha$. Show that $\sim$ is an equivalence relation.

Reflexive: Since $\mathcal{P}$ is a partition of X, $X = \cup P_\alpha$. Thus $x \in X$ implies there exists $P_\alpha \in \mathcal{P}$ such that $x \in P_\alpha$. Hence $x \sim x$.

Symmetric: Suppose $x \sim y$. Then there exists $P_\alpha \in \mathcal{P}$ such that $x, y \in P_\alpha$. Since $y, x \in P_\alpha$, $y \sim x$.

Transitive: Suppose $x \sim y$ and $y \sim z$. Then there exists $P_\alpha \in \mathcal{P}$ such that $x, y \in P_\alpha$ and there exists $P_\beta \in \mathcal{P}$ such that $y, z \in P_\beta$. Thus $y \in P_\alpha \cap P_\beta$. Since $\mathcal{P}$ is a partition and $P_\alpha \cap P_\beta \neq \emptyset$, $P_\alpha = P_\beta$. Thus $x, z \in P_\alpha$. Hence $x \sim z$.

4.) Let $\mathcal{Z}$ be the set of integers. Define the equivalence relation $\sim$ on $\mathcal{Z}$ by $x \sim y$ if and only if $5|(x-y)(xy-1)$. Show that $\sim$ is reflexive and symmetric. Use $\sim$ to partition $\mathcal{Z}$ into its equivalence classes. Make sure the sets in your partition are pairwise disjoint.

Reflexive
Claim: $x \sim x$.

$(x-x)(x^2-1) = 0$ Thus $5|(x-x)(x^2-1)$. Hence $x \sim x$.

Symmetric
Claim: $x \sim y$ implies $y \sim x$.

Suppose $x \sim y$. Then $5|(x-y)(xy-1)$. Thus $(x-y)(xy-1) = 5k$ for some integer k. Hence $(y-x)(yx-1) = 5(-k)$ where $-k$ is an integer. Thus $5|(y-x)(yx-1)$ and $y \sim x$

Equivalence classes:
Suppose $x = 5k+j$ and $y = j$. Then $(5k+j-j)[(5k+j)j-1] = (5k)[(5k+j)j-1]$.

Thus $5|(5k+j-j)((5k+j)j-1)$. Hence $5k+j \sim j$

Thus we only need to determine if the equivalence classes $[0],[1],[2],[3],[4]$ are pairwise disjoint.

$(0-k)((0)(1)-1) = k$ is divisible by 5 iff k is a multiple of 5. Thus $0 \not\sim k$ for $k = 1,2,3,4$. Hence the equivalence class $[0]$ is disjoint from $[k]$ for $k = 1,2,3,4$.

$(3-2)((3)(2)-1) = 5$ is divisible by 5. Thus $2 \sim 3$. Hence $[2] = [3]$.

$(1-2)((1)(2)-1) = -1$ is not divisible by 5. Thus $1 \not\sim 2$. Hence $[1] \cap [2] = \emptyset$.

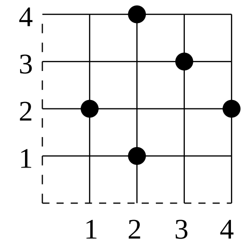
$(4-1)((4)(1)-1) = 9$ is not divisible by 5. Thus $1 \not\sim 4$. Hence $[1] \cap [4] = \emptyset$.

$(4-2)((4)(2)-1) = 14$ is not divisible by 5. Thus $2 \not\sim 4$. Hence $[2] \cap [4] = \emptyset$.

Thus $\mathcal{Z}$ is partitioned into the equivalence classes [0], [1], [2], [4] where

$[0] = \{5k \mid k \in \mathcal{Z}\}$, $[1] = \{5k+1 \mid k \in \mathcal{Z}\}$, $[2] = \{5k+2 \mid k \in \mathcal{Z}\}$, $[4] = \{5k+4 \mid k \in \mathcal{Z}\}$

5.) Let $X = \{1, 2, 3, 4\}$. Define the relation R on X by xRy if and only if $3|(2x - y)$. Draw R as a subset of $X \times X$. Determine which of the following properties hold for R (Prove it).



Is R reflexive?

No. Let $x = y = 1$. $2x - y = 2 - 1 = 1$ which is not divisible by 3. Thus $1 \not R\, 1$

Is R irreflexive?

No. Let $x = y = 3$. $2x - y = 6 - 3 = 3$ which is divisible by 3. Thus $3R3$

Is R symmetric?

Yes. We have both $1R2$ and $2R1$ as well as $2R4$ and $4R2$. Since $X = \{1, 2, 3, 4\}$, this covers all cases where $x \neq y$ and xRy.

Note R need not be symmetric if X were a larger set than $X = \{1, 2, 3, 4\}$.

Is R antisymmetric?

No. We have both $1R2$ and $2R1$, but $1 \neq 2$.

Is R transitive?

No. We have both $1R2$ and $2R4$, but we don't have $1R4$.

6.) Determine the number of 10-combinations of $\{5 \cdot a, 5 \cdot b, 5 \cdot c\}$.

Let S = 10 combinations of $\{\infty \cdot a, \infty \cdot b, \infty \cdot c\}$. Then $|S| = \binom{10+3-1}{10} = \frac{(12)(11)}{2} = 66$

Let A_1 = 10 combinations of $\{\infty \cdot a, \infty \cdot b, \infty \cdot c\}$ which contain at least 6 a's.

Let A_2 = 10 combinations of $\{\infty \cdot a, \infty \cdot b, \infty \cdot c\}$ which contain at least 6 b's.

Let A_3 = 10 combinations of $\{\infty \cdot a, \infty \cdot b, \infty \cdot c\}$ which contain at least 6 c's.

$|A_1|$ = # of 10 combinations of $\{\infty \cdot a, \infty \cdot b, \infty \cdot c\}$ which contain at least 6 a's
= # of 10 - 6 = 4 combinations of $\{\infty \cdot a, \infty \cdot b, \infty \cdot c\} = \binom{4+3-1}{4} = \frac{(6)(5)}{2} = 15$

Similarly $|A_2| = |A_3| = 15$.

$|A_1 \cap A_2|$ = # of 10 combinations of $\{\infty \cdot a, \infty \cdot b, \infty \cdot c\}$ which contain at least 6 a's and at least 6 b's = 0.

Similarly $|A_1 \cap A_3| = |A_2 \cap A_3| = |A_1 \cap A_2 \cap A_3| = 0$

Thus the number of 10-combinations of $\{5 \cdot a, 5 \cdot b, 5 \cdot c\} =$
$|S| - \Sigma|A_i| + \Sigma|A_i \cap A_j| - |A_1 \cap A_2 \cap A_3| = 66 - 3(15) = 66 - 45 = 21.$

7.) Prove that $(x + y + z)^n = \Sigma \binom{n}{n_1\ n_2\ n_3} x^{n_1}y^{n_2}z^{n_3}$.

When multiplying out $(x+y+z)^n$, we obtain terms of the form $x^{n_1}y^{n_2}z^{n_3}$ where $n_1+n_2+n_3 = n$ as each of the n factors of $(x+y+z)^n$ contributes an x, y, or z to each term of $(x+y+z)^n$.

Note that each term of $(x+y+z)^n$ corresponds to a permutation of the multiset $\{\infty \cdot x, \infty \cdot y, \infty \cdot z\}$ where if the permutation contains n_1 x's, n_2 y's, and n_3 z's, then $n_1 + n_2 + n_3 = n$.

Thus the coefficient of $x^{n_1}y^{n_2}z^{n_3}$ = the number of terms where the permutation contains n_1 x's, n_2 y's and n_3 z's and $n_1 + n_2 + n_3 = n$.

The number of permutation which contain n_1 x's, n_2 y's and n_3 z's where $n_1 + n_2 + n_3 = n$ is $\binom{n}{n_1\ n_2\ n_3}$.

Thus $(x + y + z)^n = \Sigma \binom{n}{n_1\ n_2\ n_3} x^{n_1}y^{n_2}z^{n_3}$.

Alternate proof. When multiplying out $(x + y + z)^n$, we obtain terms of the form $x^{n_1}y^{n_2}z^{n_3}$ where $n_1 + n_2 + n_3 = n$ as each of the n factors of $(x + y + z)^n$ contributes an x, y, or z to each term of $(x + y + z)^n$. To form a term $x^{n_1}y^{n_2}z^{n_3}$, we

(1) need to choose n_1 x's from the n x's appearing in $(x + y + z)^n$

(2) from the remaining $n - n_1$ factors from which x was not chosen, we need to choose n_2 y's

(3) choose all n_3 z's from the remaining $n - n_1 - n_2 = n_3$ factors from which neither x nor y was chosen.

The number of ways to choose n_1 x's from n x's is $\binom{n}{n_1}$.

The number of ways to choose n_2 y's from $n - n_1$ y's is $\binom{n - n_1}{n_2}$.

The number of ways to choose n_3 z's from $n - n_1 - n_2$ z's is $\binom{n - n_1 - n_2}{n_3}$.

Thus the coefficient of $x^{n_1}y^{n_2}z^{n_3}$ is $\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_3}{n_3}$

$= \frac{n!}{n_1!(n-n_1)!}\frac{(n-n_1)!}{n_2!(n-n_1-n_2)!}\frac{(n-n_1-n_2)!}{n_3!(n-n_1-n_2-n_3)!} = \frac{n!}{n_1!n_2!n_3!(0)!}$

Thus $(x+y+z)^n = \Sigma \binom{n}{n_1\ n_2\ n_3} x^{n_1}y^{n_2}z^{n_3}$.

8a.) Use the binomial theorem to prove that $2^n = \Sigma_{k=0}^n \binom{n}{k}$.

Binomial theorem: $(x+y)^n = \Sigma_{k=0}^n \binom{n}{k} x^k y^{n-k}$.

Let $x = 1$, $y = 1$. Then $2^n = (1+1)^n = \Sigma_{k=0}^n \binom{n}{k} (1)^k(1)^{n-k} = \Sigma_{k=0}^n \binom{n}{k}$.

8b.) Generalize to find the sum $\Sigma_{k=0}^n \binom{n}{k} r^k$.

Let $x = r$, $y = 1$. Then $(r+1)^n = \Sigma_{k=0}^n \binom{n}{k} (r)^k(1)^{n-k} = \Sigma_{k=0}^n \binom{n}{k} r^k$.