6.6 Mobius inversions

Let X be a finite set.

Let $\mathcal{F} = \{f : X \times X \to \mathcal{R} \mid \text{ if } f(x,y) \neq 0, \text{ then } x \leq y\}$

Define the operation * on $\mathcal{F}$ by

$$f * g = \begin{cases} \Sigma_{\{z \ \mid \ x \leq z \leq y\}} f(x,z)g(z,y) & \text{if } x \leq y \\ 0 & \text{otherwise} \end{cases}$$

Note $*$ is associative: $f * (g * h) = (f * g) * h$

Let $\delta = \begin{cases} 1 & \text{x = y} \\ 0 & \text{otherwise} \end{cases}$

Note δ acts as the identity for *: $f * \delta = \delta * f = f$

If $f(x,x) \neq 0$ for all $x \in X$, then f is invertible: There exist f^{-1} such that $f * f^{-1} = f^{-1} * f = \delta$. In this case,

$f^{-1}(x,x) = \frac{1}{f(x,x)}$

$f^{-1}(x,y) = -\Sigma_{\{z \ : \ x \leq z < y\}} f^{-1}(x,z) \frac{f(z,y)}{f(y,y)}$ for $x < y$,

$f^{-1}(x,y) = 0$ if $x \not\leq y$.