HW 4 part 1

Thm 6.4 (Inverse Function Theorem): Suppose $F : W^{open} \subset \mathbf{R}^n \to \mathbf{R}^n \in C^r$. Suppose for $\mathbf{a} \in W$, $det(DF(a)) \neq 0$. Then there exists U such that $\mathbf{a} \in U^{open}$, $V = F(U)$ is open, and $F : U \to V$ is a C^r-diffeomorphism. Moreover, for $\mathbf{x} \in U$ and $\mathbf{y} = F(\mathbf{x})$, $DF^{-1}_{\mathbf{y}} = (DF_{\mathbf{x}})^{-1}$

Proof: Recall from HW 3 part 2, we need to show (1.) F is 1:1 on some open set containing $\mathbf{0}$, (2.) F^{-1} is continuous, and (3.) $\frac{||F^{-1}(\mathbf{y})-F^{-1}(\mathbf{d})||}{||\mathbf{y}-\mathbf{d}||}$ is bounded for $\mathbf{y}, \mathbf{d}$ in some neighorhood of $\mathbf{0}$.

You may either prove the above by continuing method 1 in HW 3 part 2 (see Spivak) or by the following method (Boothby).

Let $G(\mathbf{x}) = \mathbf{x} - F(\mathbf{x})$. $G(\mathbf{0}) =$ _______, $DG(\mathbf{0}) =$ _______, $\frac{\partial g_i}{\partial x_j}(\mathbf{0}) =$ _______,

Since $F \in C^r$, $G \in C^r$ and $\frac{\partial g_i}{\partial x_j}$ is continuous for all i, j.

1.) Use Theorem 2.2 to show that there exists $r > 0$ such that DF is nonsingular on the closed ball $\overline{B}_{2r}(\mathbf{0})$ and for $\mathbf{x}_1, \mathbf{x}_2 \in \overline{B}_r(\mathbf{0})$, $||G(\mathbf{x}_1) - G(\mathbf{x}_2)|| \leq \frac{1}{2}||\mathbf{x}_1 - \mathbf{x}_2||$ (Eqn *)

$||G(\mathbf{x}_1) - G(\mathbf{x}_2)|| = ||\mathbf{x}_1 - F(\mathbf{x}_1) - \mathbf{x}_2 + F(\mathbf{x}_2)||$

Thus $||\mathbf{x}_1 - \mathbf{x}_2|| - ||F(\mathbf{x}_1) - F(\mathbf{x}_2)|| \leq ||\mathbf{x}_1 - F(\mathbf{x}_1) - \mathbf{x}_2 + F(\mathbf{x}_2)|| \leq \frac{1}{2}||\mathbf{x}_1 - \mathbf{x}_2||$

Hence $\frac{1}{2}||\mathbf{x}_1 - \mathbf{x}_2|| \leq ||F(\mathbf{x}_1) - F(\mathbf{x}_2)||$ and thus

$$||\mathbf{x}_1 - \mathbf{x}_2|| \leq 2||F(\mathbf{x}_1) - F(\mathbf{x}_2)||(eqn **)$$

By Eqn (*), $||G(\mathbf{x}_1) - G(\mathbf{x}_2)|| \leq \frac{1}{2}||\mathbf{x}_1 - \mathbf{x}_2||$ for all $\mathbf{x}_1, \mathbf{x}_2 \in \overline{B}_{2r}(\mathbf{0})$,

Thus for $\mathbf{x} \in \overline{B}_{2r}(\mathbf{0})$, $||G(\mathbf{x})|| = ||G(\mathbf{x}) - G(\mathbf{0})|| \leq \frac{1}{2}||\mathbf{x} - \mathbf{0}|| = \frac{1}{2}||\mathbf{x}||$

Let $\mathbf{y} \in \overline{B}_{\frac{r}{2}}(\mathbf{0})$ and let $T_{\mathbf{y}}(\mathbf{x}) = \mathbf{y} + \mathbf{x} - F(\mathbf{x})$.

Suppose $\mathbf{x} \in \overline{B}_r(\mathbf{0})$. Then $||T_{\mathbf{y}}(\mathbf{x}) - \mathbf{0}|| = ||\mathbf{y} + \mathbf{x} - F(\mathbf{x})|| = ||\mathbf{y} + G(\mathbf{x})|| \leq ||\mathbf{y}|| + ||G(\mathbf{x})|| \leq ||\mathbf{y}|| + \frac{1}{2}||\mathbf{x}|| \leq \frac{r}{2} + \frac{r}{2} = r$. Thus $T_{\mathbf{y}}(\mathbf{x}) \in \overline{B}_r(\mathbf{0})$.

Hence $T_{\mathbf{y}} : \overline{B}_r(\mathbf{0}) \to \overline{B}_r(\mathbf{0})$.

2.) Let $T_{\mathbf{y}} : \overline{B}_r(\mathbf{0}) \to \overline{B}_r(\mathbf{0})$, $T_{\mathbf{y}}(\mathbf{x}) = \mathbf{y} + \mathbf{x} - F(\mathbf{x})$. Show that $T_{\mathbf{y}}$ has a unique fixed point iff there is a unique $\mathbf{x} \in \overline{B}_r(\mathbf{0})$ such that $F(\mathbf{x}) = \mathbf{y}$

Thus if for each $y \in \overline{B}_{\frac{r}{2}}(\mathbf{0})$, the function $T_{\mathbf{y}}(\mathbf{x})$ has a unique fixed point, then F^{-1} exists on $\overline{B}_{\frac{r}{2}}(\mathbf{0})$

Claim: If $y \in \overline{B}_{\frac{r}{2}}(\mathbf{0})$, $T_{\mathbf{y}} : \overline{B}_r(\mathbf{0}) \to \overline{B}_r(\mathbf{0})$ has a unique fixed point

We will show that $T_{\mathbf{y}}$ is a contraction:

$||T_{\mathbf{y}}(\mathbf{x}_1) - T_{\mathbf{y}}(\mathbf{x}_2)|| = ||\mathbf{y} + \mathbf{x}_1 - F(\mathbf{x}_1) - (\mathbf{y} + \mathbf{x}_2 - F(\mathbf{x}_2))|| = ||\mathbf{x}_1 - F(\mathbf{x}_1) - (\mathbf{x}_2 - F(\mathbf{x}_2))|| = ||G(\mathbf{x}_1) - G(\mathbf{x}_2)|| \leq \frac{1}{2}||\mathbf{x}_1 - \mathbf{x}_2||$ by eqn (*).

Thus $T_{\mathbf{y}}$ is a contraction. $\overline{B}_r(\mathbf{0})$ is a complete metric space. Thus $T_{\mathbf{y}}$ has a unique fixed point by the Contracting Mapping Theorem.

Let $U = F^{-1}(B_{\frac{r}{2}}(\mathbf{0}))$. Since F is continuous, U is open. Let $V = B_{\frac{r}{2}}(\mathbf{0})$.

3.) Use eqn (**) to show that $F^{-1} : V \to U$ is continuous and that $\frac{||F^{-1}(\mathbf{y}) - F^{-1}(\mathbf{d})||}{||\mathbf{y} - \mathbf{d}||}$ is bounded for $\mathbf{y}, \mathbf{d}$ in some neighorhood of $\mathbf{0}$.