Point Set Topology

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Theorem:

The image of a connected space under a continuous map is connected.

Proof:

Let $f : X \to Y$ be a continuous function. We wish to show that:

$$X \text{ is conntected} \Rightarrow f(X) \text{ is conntected}$$

The contrapositive of this statement is:

$$f(X) \text{ is not conntected} \Rightarrow X \text{ is not conntected}$$

which by definition is equivalent to:

$$\exists \text{ a separation of } f(X) \Rightarrow \exists \text{ a separation of } X$$

Suppose there exists a separation of $f(X)$, i.e. $\exists A$ and B disjoint, nonempty, open sets such that $A \cup B = f(X)$.

Note that the function $g : X \to f(X)$ is surjective and continuous, since $f : X \to Y$ is continuous by assumption.

Therefore $g^{-1}(A)$ and $g^{-1}(B)$ are disjoint, nonempty, open sets such that $g^{-1}(A) \cup g^{-1}(B) = X$. (They are nonempty because g is surjective and open because g is continuous).

Therefore $g^{-1}(A)$ and $g^{-1}(B)$ form a separation of X.

QED