Point Set Topology

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Theorem:

The image of a compact space under a continuous map is compact.

Proof:

Let $f : X \to Y$ be a continuous function. We wish to show that:

$$X \text{ is compact} \Rightarrow f(X) \text{ is compact}$$

Let $\mathcal{A}$ be an open cover of $f(X)$ (where all the sets of $\mathcal{A}$ are open in Y).
Then the collection

$$\{f^{-1}(A) | A \in \mathcal{A}\}$$

is an open cover of X. Note that these sets are open because f is continuous by assumption.
Since X is compact, there exists a finite subcover:

$$\{f^{-1}(A_1)\ ,\ f^{-1}(A_2)\ ,\ \dots\ ,\ f^{-1}(A_n)\}$$

which covers X. Therefore the collection

$$\{A_1\ ,\ A_2\ ,\ \dots\ ,\ A_n\}$$

forms a finite subcover of $\mathcal{A}$ which covers $f(X)$.
Therefore $f(X)$ is compact.
QED