5.5 Series Solutions Near a Regular Singular Point, Part I

Theorem 5.3.1: If $p(x)$ and$q(x)$ are analytic at x_0 (i.e., x_0 is an ordinary point of the ODE $y''+p(x)y'+q(x)y=0$), then the general solution to this ODE is

$$y = \Sigma_{n=0}^{\infty} a_n(x-x_0)^n = a_0\phi_0(x) + a_1\phi_1(x)$$

where ϕ_i are power series solutions that are analytic at x_0. The solutions ϕ_0, ϕ_1 form a fundamental set of solutions. The radius of convergence for each of these series solutions is at least as large as the minimum radii of convergence of the series for $\frac{Q}{P}$ and $\frac{R}{P}$.

If you prefer a power series expansion about 0, use u-substitution: let $u = x - x_0$. Then $p(u+x_0)$ and $q(u+x_0)$ are analytic at 0

(Semi-failed) attempt to transform 5.5 problem into 5.4 problem:

5.5: $y'' + p(x)y' + q(x)y = 0$

$x^2y'' + x^2p(x)y' + x^2q(x)y = 0$

$x^2y'' + x[xp(x)]y' + [x^2q(x)]y = 0$ where $xp(x)$ and $x^2q(x)$ are functions of x.

5.4: $x^2y'' + \alpha xy' + \beta y = 0$ **where** α, β **are constants.**

Combine 5.3/5.4 methods.

Defn: x_0 is a *regular singular value* if x_0 is a singular value and $xp(x)$ and $x^2q(x)$ are analytic at x_0. A singular value which is not regular is called *irregular*.

Examples:

$y'' + \frac{y'}{x} + \frac{y}{x^2} = 0$, regular singular value: $x = 0$.

$y'' + \frac{y'}{x^2} + \frac{y}{x} = 0$, irregular singular value: $x = 0$.

$y'' + y' + \frac{y}{x^3} = 0$, irregular singular value: $x = 0$.

If $p(x)$ and $q(x)$ are rational functions, then $xp(x)$ and $x^2q(x)$ are analytic iff $lim_{x\to 0}xp(x)$ and iff $lim_{x\to 0}x^2q(x)$ are finite. (i.e., after reducing fractions, x is not in the denominator.

Ex: $p(x) = \frac{1}{x}$ implies $xp(x) = \frac{x}{x} = 1$

Ex: $p(x) = \frac{1}{x^2}$ implies $xp(x) = \frac{x}{x^2} = \frac{1}{x}$

If $x_0 = 0$ is a regular singular value of the linear homogeneous DE, $x^2y'' + x[xp(x)]y' + x^2q(x)y = 0$ (*), then

$xp(x) = \Sigma_{n=0}^{\infty} p_nx^n$ and $x^2q(x) = \Sigma_{n=0}^{\infty} q_nx^n$ for constants p_n, q_n.

If $y = x^r\Sigma_{n=0}^{\infty} a_nx^n = \Sigma_{n=0}^{\infty} a_nx^{n+r}$ is a solution to (*) where $r \neq 0$.

$y' = \Sigma_{n=0}^{\infty}(n+r)a_nx^{n+r-1}$ and $y'' = \Sigma_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r-2}$

$$x^2\Sigma_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r-2} + x[xp(x)]\Sigma_{n=0}^{\infty}(n+r)a_nx^{n+r-1} + [x^2q(x)]\Sigma_{n=0}^{\infty}a_nx^{n+r}$$

$$\Sigma_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r} + [xp(x)]\Sigma_{n=0}^{\infty}(n+r)a_nx^{n+r} + [x^2q(x)]\Sigma_{n=0}^{\infty}a_nx^{n+r}$$

$$\Sigma_{n=0}^{\infty}(n+r)(n+r-1)a_nx^{n+r} + \left(\Sigma_{n=0}^{\infty}p_nx^n\right)\left(\Sigma_{n=0}^{\infty}(n+r)a_nx^{n+r}\right) + \left(\Sigma_{n=0}^{\infty}q_nx^n\right)\left(\Sigma_{n=0}^{\infty}a_nx^{n+r}\right)$$

Thus the coefficient of x^r is $r(r-1)a_0 + p_0ra_0 + q_0a_0 = 0$

We can take $a_0 \neq 0$. Thus $r(r-1) + p_0r + q_0 = 0$

Thus we can solve for r using the quadratic formula.

Case 1: $r_1 > r_2$ both real and $r_1 - r_2$ is not an integer.
Case 2: $r_1 > r_2$ both real and $r_1 - r_2 = p$, p an integer.
Case 3: one repeated root.
Case 4: two complex roots.