5.5: Solve $x^2y'' - x(2+x)y' + (2+x^2)y = 0$

$p(x) = -\frac{x(2+x)}{x^2} = -\frac{2+x}{x}$. Thus $x_0 = 0$ is a singular value.

$q(x) = \frac{2+x^2}{x^2}$ also implies $x_0 = 0$ is a singular value.

$xp(x) = -(2+x)$ and $x^2q(x) = 2+x^2$. Thus $x_0 = 0$ is a regular singular value.

Suppose $y = \sum_{n=0}^{\infty} a_n x^{n+r}$ is a solution. WLOG assume $a_0 \neq 0$ (otherwise one can reindex the summation).

Then $y' = \sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}$ and $y'' = \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}$

$x^2y'' - x(2+x)y' + (2+x^2)y$

$= x^2 \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} - (2x+x^2)\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} + (2+x^2)\sum_{n=0}^{\infty} a_n x^{n+r}$

$$= \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} - \sum_{n=0}^{\infty} 2(n+r)a_n x^{n+r} - \sum_{n=0}^{\infty}(n+r)a_n x^{n+r+1} + \sum_{n=0}^{\infty} 2a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+2}$$

$$= \sum_{n=0}^{\infty}[(n+r)(n+r-1) - 2(n+r) + 2]a_n x^{n+r} - \Sigma_{n=1}^{\infty}(n+r-1)a_{n-1}x^{n+r} + \Sigma_{n=2}^{\infty}a_{n-2}x^{n+r}$$

$$= [r(r-1) - 2r + 2]a_0x^r + [(1+r)r - 2(1+r) + 2]a_1x^{r+1} - ra_0x^{r+1} + \Sigma_{n=2}^{\infty}[(n+r)(n+r-1) - 2(n+r) + 2]a_n x^{n+r} - \Sigma_{n=2}^{\infty}(n+r-1)a_{n-1}x^{n+r} + \Sigma_{n=2}^{\infty}a_{n-2}x^{n+r}$$

$$= [r(r-1) - 2r + 2]a_0x^r + \big([(1+r)r - 2(1+r) + 2]a_1 - ra_0\big)x^{r+1} + \Sigma_{n=2}^{\infty}\big([(n+r)(n+r-1) - 2(n+r) + 2]a_n - (n+r-1)a_{n-1} + a_{n-2}\big)x^{n+r}$$

$$= [r^2 - r - 2r + 2]a_0x^r + \big([r + r^2 - 2 - 2r + 2]a_1 - ra_0\big)x^{r+1} + \Sigma_{n=2}^{\infty}\big([(n+r)(n+r-3) + 2]a_n - (n+r-1)a_{n-1} + a_{n-2}\big)x^{n+r}$$

$$= [r^2 - 3r + 2]a_0x^r + \big([r^2 - r]a_1 - ra_0\big)x^{r+1} + \Sigma_{n=2}^{\infty}\big([n^2 + 2rn + r^2 - 3n - 3r + 2]a_n - (n+r-1)a_{n-1} + a_{n-2}\big)x^{n+r} = 0$$

Set all coefficients = 0:

Since $a_0 \neq 0$, $r^2 - 3r + 2 = (r-2)(r-1) = 0$ implies $r = 1, 2$.

$r^2 - 3r + 2 = 0$ is the *indicial equation*

$[r^2 - r]a_1 = ra_0$ implies $(r-1)a_1 = a_0$. Thus if $r = 1$, $a_0 = 0$, a contradiction. If $r = 2$, $a_1 = a_0$

$$[n^2+2rn+r^2-3n-3r+2]a_n-(n+r-1)a_{n-1}+a_{n-2} = [n^2+2rn-3n]a_n-(n+r-1)a_{n-1}+a_{n-2} = 0$$

$$a_n = \frac{(n+r-1)a_{n-1}-a_{n-2}}{n^2+2rn-3n} = \frac{(n+1)a_{n-1}-a_{n-2}}{n^2+4n-3n} = \frac{(n+1)a_{n-1}-a_{n-2}}{n^2+n} = \frac{(n+1)a_{n-1}-a_{n-2}}{n(n+1)}$$

$$a_2 = \frac{3a_1-a_0}{6} = \frac{3a_0-a_0}{6} = \frac{2a_0}{6} = \frac{a_0}{3}$$

$$a_3 = \frac{4a_2-a_1}{(3)(4)} = \frac{4(\frac{a_0}{3})-a_0}{(3)(4)} = \frac{4a_0-3a_0}{(3)^2(4)} = \frac{a_0}{(3)^2(4)}$$

$$a_4 = \frac{5a_3-a_2}{(4)(5)} = \frac{(5\frac{a_0}{(3)^2(4)})-(\frac{a_0}{3})}{(4)(5)} = \frac{5a_0-3(4)a_0}{3^2(4)^2(5)} = \frac{7a_0}{3^2(4)^2(5)}$$

$$a_5 = \frac{6a_4-a_3}{(5)(6)} = \frac{6(\frac{7a_0}{3^2(4)^2(5)})-(\frac{a_0}{(3)^2(4)})}{(5)(6)} = \frac{6(7a_0)-(20a_0)}{(3)^2(4)^2(5)^2(6)} = \frac{22a_0}{(3)^2(4)^2(5)^2(6)}$$

$$a_6 = \frac{7a_5-a_4}{(6)(7)} = \frac{7(\frac{22a_0}{(3)^2(4)^2(5)^2(6)})-\frac{7a_0}{3^2(4)^2(5)}}{(6)(7)} = \frac{7(22a_0)-30(7a_0)}{(3)^2(4)^2(5)^2(6)^2(7)} = \frac{-56a_0}{(3)^2(4)^2(5)^2(6)^2(7)}$$

$$a_7 = \frac{8a_6-a_5}{(7)(8)} = \frac{8(\frac{-56a_0}{(3)^2(4)^2(5)^2(6)^2(7)})-\frac{22a_0}{(3)^2(4)^2(5)^2(6)}}{(7)(8)} = \frac{8(-56a_0)-42*22a_0}{(3)^2(4)^2(5)^2(6)^2(7)^2(8)} = \frac{-1372a_0}{(3)^2(4)^2(5)^2(6)^2(7)^2(8)}$$

$$y = x^2(a_0+a_0x+\frac{a_0}{3}x^2+\frac{a_0}{(3)^2(4)}x^3+\frac{7a_0}{3^2(4)^2(5)}x^4+\frac{22a_0}{(3)^2(4)^2(5)^2(6)}x^5+\frac{-56a_0}{(3)^2(4)^2(5)^2(6)^2(7)}x^6+\frac{-1372a_0}{(3)^2(4)^2(5)^2(6)^2(7)^2(8)}x^7+...) \blacksquare$$