5.1 Review of Power Series.

Definition: $\Sigma_{n=0}^{\infty} a_n(x-x_0)^n = lim_{n\to\infty}\Sigma_{n=0}^{k} a_n(x-x_0)^n$

Taylor's Theorem

Suppose f has $n+1$ continuous derivatives on an open interval containing a. Then for each x in the interval,

$$f(x) = \left[\sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k\right] + R_{n+1}(x)$$

where the error term $R_{n+1}(x)$ satisfies $R_{n+1}(x) = \dfrac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$ for some c between a and x.

The *infinite* Taylor series converges to f,

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k \text{ if and only if } \lim_{n\to\infty} R_n(x) = 0.$$

Defn: The function f is said to be **analytic** at a is its Taylor series expansion about $x = a$ has a positive radius of convergence.

1.) $\Sigma_{n=0}^{\infty} a_n(x-x_0)^n$ converges at the point x if and only if $lim_{n\to\infty}\Sigma_{n=0}^{k} a_n(x-x_0)^n$ exists at the point x.

2.) $\Sigma_{n=0}^{\infty} a_n(x-x_0)^n$ converges absolutely at the point x if and only if $\Sigma_{n=0}^{\infty}|a_n||x-x_0|^n$ converges at the point x

If a series converges absolutely, then it also converges.

3.) Ratio test for absolute convergence:

Let $L = lim_{n\to\infty}|\frac{a_{n+1}}{a_n}|$

$lim_{n\to\infty}|\frac{a_{n+1}(x-x_0)^{n+1}}{a_n(x-x_0)^n}| = |x-x_0|lim_{n\to\infty}|\frac{a_{n+1}}{a_n}| = |x-x_0|L$

The power series converges at the value x if $|x-x_0| < \frac{1}{L}$

The power series diverges at the value x if $|x-x_0| > \frac{1}{L}$

The ratio test give no info at the value x if $|x-x_0| = \frac{1}{L}$

Note $\frac{1}{L}$ is the **radius of convergence**.