SOLVING UNORIENTED TANGLE EQUATIONS INVOLVING 4-PLATS

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Revised

ABSTRACT

The system of unoriented tangle equations $N(U + \frac{f_1}{g_1}) = K_1$ and $N(U + \frac{f_2}{g_2}) = K_2$ is completely solved for the tangles U and $\frac{f_2}{g_2}$ as a function of $\frac{f_1}{g_1}$ where K_1 and K_2 are 4-plats, and $\frac{f_1}{g_1}$ and $\frac{f_2}{g_2}$ rational tangles such that $|f_1 g_2 - g_1 f_2| > 1$.

1. Introduction

Given two knots/links, K_1, K_2, it is of much biological interest to solve tangle equations of the form $N(U + P) = K_1$ and $N(U + R) = K_2$ [8, 17, 9, 2, 4]. In many biological applications the knots/links involved are 4-plats, and in many of these cases, it is possible to prove that U is ambient isotopic to a sum of rational tangles and P and R are rational tangles [11, 8, 7, 9, 4, 12, 15]. In this situation it is possible to list all solutions for U and R as a function of P.

In section 2 a brief introduction to tangles is given. The system of unoriented tangle equations $N(U+P) = K_1$ and $N(U+R) = K_2$ is solved in section 3 where P is the zero tangle, R is rational, U is a generalized Montesinos tangle, and K_1 and K_2 are 4-plats. Equivalent moves are discussed in sections 4 and 5. The results of section 3 are extended in section 5 to solve the system of equations, $N(U+P) = K_1$ and $N(U + R) = K_2$, where P and R are arbitrary rational tangles. These results are summarized in theorem 6.

These theorems are a consequence of the cyclic surgery theorem [3], a theorem of Ernst's [8], and much tangle manipulation. Since it is necessary to implement these theorems for biological applications, a program which performs these calculations is available at the following URL:

http://www.math.uiowa.edu/∼idarcy/PROG/comput.html

This subroutine will also be included in Rob Scharein's KnotPlot which is available at www.KnotPlot.com.

2. Tangle Notation

A 2-string tangle is a pair (B^3, t) where B^3 is a 3-dimensional ball, $\{x \in \mathbf{R^3} : |x| \leq 1\}$, and t is a pair of arcs and a finite number of circles properly embedded in B^3. Some examples of tangles are shown in Fig. 1. The four endpoints of the arcs will be fixed at $NW = (e^{5i\pi/4}, 0)$, $NE = (e^{i\pi/4}, 0)$, $SW = (e^{-5i\pi/4}, 0)$, $SE = (e^{-i\pi/4}, 0)$. Two tangles are equivalent if they are ambient isotopic keeping the boundary of B^3 fixed. A tangle is rational if it is ambient isotopic to the zero tangle where the boundary of B^3 need not be fixed.

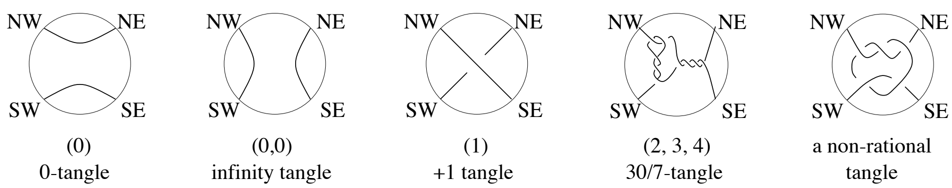

Fig. 1. Tangle examples

A rational tangle can be constructed from the zero tangle by alternating between rotating the NE and SE endpoints and the SW and SE endpoints as shown in Fig. 2. The tangle obtained can be represented by the sequence of numbers denoting the number of half twists where the numbers alternate between representing vertical twists versus horizontal twists with the last number always representing horizontal twists. A tangle $(x_1, ..., x_n)$ is uniquely identified by its continued fraction, $x_n + \frac{1}{x_{n-1} + \dots \frac{1}{x_1}}$. Thus, two tangles are equivalent if and only if their continued fractions are the same. For example the tangles (2, 3, 4) and (-2, -4, 1, 3) shown in Fig. 3 are equivalent since $4 + \frac{1}{3+\frac{1}{2}} = \frac{30}{7} = 3 + \frac{1}{1+\frac{1}{-4+\frac{1}{-2}}}$.

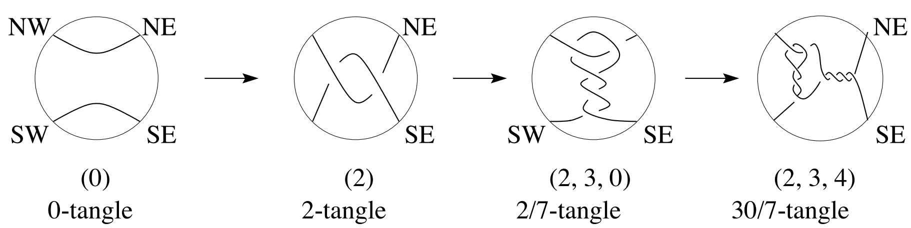

Fig. 2. Drawing the (2,3,4) tangle

Tangles can be added (Fig. 4). The circle product of A and $C = (c_1, ..., c_n)$, is shown in Fig. 5 when n is even and in Fig. 6 when n is odd. A generalized Montesinos tangle or generalized M-tangle is a tangle of the form $(A_1 + ... + A_n) \circ C$ where $A_i, 1 \leq i \leq n$, and C are rational tangles. Note that a generalized M-tangle is a rational tangle if all but at most one of the A_i' s are integral. In particular, the sum of two rational tangles is a rational tangle if and only if one of the tangles is integral. In this case the tangle $\frac{a}{b} + i$ equals the tangle $\frac{a+bi}{b}$. Also

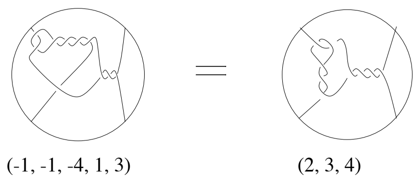

Fig. 3. The $\frac{30}{7}$-tangle

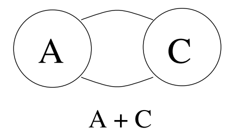

Fig. 4. Adding tangles

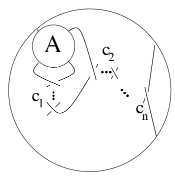

Fig. 5. $A \circ (c_1, ..., c_n)$, n even

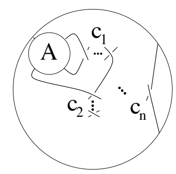

Fig. 6. $A \circ (c_1, ..., c_n)$, n odd

since $\frac{a}{b} + \frac{c}{d} = \frac{a}{b} - i + i + \frac{c}{d} = \frac{a-bi}{b} + \frac{c+di}{d}$, the tangle $\frac{a}{b} + \frac{c}{d}$ is the same as the tangle $\frac{a-bi}{b} + \frac{c+di}{d}$.

A knot or link is formed by taking the numerator closure of a tangle as shown in Fig. 7. A 4-plat (or 2-bridge or rational knot/link) is a knot or link which can be written as the numerator closure of a rational tangle. Two unoriented 4-plats $N(a_1/b_1)$ and $N(a_2/b_2)$, $a_i \geq 0$, are the same if and only if $a_1 = a_2$ and $b_1 b_2^{\pm 1} \cong 1$ (mod a_1) [1].

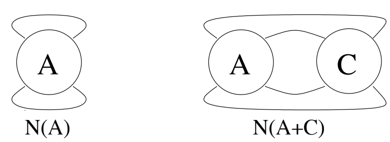

Fig. 7. Numerator closure

If $N(U + P) = K_1$ and $N(U + R) = K_2$, then K_2 is said to have been obtained from K_1 by a (P, R) move. In section 5 it will be shown that $N(U + \frac{f_1}{g_1}) = K_1$ and $N(U + \frac{f_2}{g_2}) = K_2$ for some tangle U if and only if $N(U' + \frac{0}{1}) = K_1$ and $N(U' + \frac{t}{w}) = K_2$ for some tangle U' where $\frac{t}{w} = \frac{g_1 f_2 - g_2 f_1}{e_1 g_2 - i_1 f_2}$, e_1, i_1 are integers such that $g_1 e_1 - f_1 i_1 = 1$ (theorem 5). Thus an $(\frac{f_1}{g_1}, \frac{f_2}{g_2})$ move will be said to be equivalent to a $(\frac{0}{1}, \frac{g_1 f_2 - g_2 f_1}{e_1 g_2 - i_1 f_2})$ move.

3. Solving the Unoriented Equations $N(U + \frac{0}{1}) = N(\frac{a}{b})$, $N(U + \frac{t}{w}) = N(\frac{z}{v})$

In theorem 3 the unoriented system of equations $N(U + \frac{0}{1}) = N(\frac{a}{b})$, $N(U + \frac{t}{w}) = N(\frac{z}{v})$ will be solved assuming that U is a generalized M-tangle. Fortunately, the following two theorems show that in most cases U must be a generalized M-tangle.

Theorem 1. (Ernst, [7]) *If $N(U + \frac{0}{1}) = N(\frac{a}{b})$ and $N(U + \frac{t}{w}) = N(\frac{z}{v})$ and if $|t| > 1$, then U is a generalized M-tangle or equivalently, $U = A \circ C$ where A is a finite sum of rational tangles and C is a rational tangle.*

Proof. By the cyclic surgery theorem [3], the double branch cover of U is Seifert fibered. Since this fibration can be extended to a generalized Seifert fibration of $L(a,b)$, the orbit surface of a Seifert fibration of U can be taken to be a disc. By Ernst [7] (see also [4]), U is a generalized M-tangle $\Box$.

Theorem 2. (Hirasawa and Shimokawa, [12]) *If* $N(U+\frac{0}{1}) = N(\frac{1}{0})$ *and* $N(U+\frac{1}{w}) = N(\frac{2k}{1})$, *then* U *is rational.*

The following lemmas are useful in calculations.

Lemma 1. *N(A + C) = N(C + A) where A and C are arbitrary tangles.*

Lemma 2. $N(A \circ (c_1, ..., c_n) + B) = N(A + B \circ (c_n, ..., c_1))$ *if* A *or* B *is rational,* n *odd.*

Lemma 3. [8] $N(\frac{j}{p} + \frac{t}{w}) = N(\frac{jw+pt}{dw+qt})$ *where* d *and* q *are any integers such that* pd - qj = 1.

Lemma 4. If $N(\frac{j}{p} + \frac{f}{g}) = N(\frac{a}{b})$, then $\frac{f}{g} = \frac{da-jb'}{pb'-qa}$ for some integers d, q, and b' such that $pd - qj = 1$, $b'b^{\pm 1} = 1 \bmod a$.

Proof. $N(\frac{j}{p}+\frac{f}{g}) = N(\frac{jg+pf}{dg+qf}) = N(\frac{a}{b})$ where d and q are any integers such that $pd - qj = 1$. Since $\frac{f}{g} = \frac{-f}{-g}$ and $N(\frac{j}{p} + \frac{-f}{-g}) = N(\frac{-jg-pf}{-dg-qf}) = N(\frac{jg+pf}{dg+qf})$, $jg+pf$ will be taken to have the same sign as a. Thus $jg+pf = a$ and $dg+qf = b'$ where $b'b^{\pm 1} = 1 \; mod \; a$. Multiplying the first equation by d, the second equation by j, and solving for f results in $f = (pd-qj)f = da - jb'$. Multiplying the first equation by q, the second equation by p, and solving for g results in $g = (pd-qj)g = pb' - qa$ $\Box$.

Theorem 3. $N(U+\frac{0}{1}) = N(\frac{a}{b})$ *and* $N(U+\frac{t}{w}) = N(\frac{z}{v})$ *where* $N(\frac{a}{b})$ *and* $N(\frac{z}{v})$ *are unoriented 4-plats and* U *is a generalized M-tangle if and only if the following hold:*

(a) If $w \not\cong \pm 1 \; mod \; t$, *then there exists an integer,* b' *such that* $b'b^{\pm 1} = 1 \; mod \; a$, *and for any integers* x *and* y *such that* $b'x - ay = 1$,

$$N(\frac{z}{v}) = N(\frac{tb' + wa}{ty + wx}) \tag{3.1}$$

In this case, $U = \frac{a}{b'}$ *for all* b' *satisfying the above.*

(b) If $w \cong \pm 1 \; mod \; t$, *then there exists relatively prime integers,* p *and* q, *where* p *may be chosen to be positive, such that*

$$N(\frac{z}{v}) = N(\frac{tp(pb-qa) \pm a}{tq(pb-qa) \pm b}) \tag{3.2}$$

In this case, the solutions for U *are* $(\frac{da-jb}{pb-qa} + \frac{j}{p}) \circ (h, 0)$ *and* $(\frac{j}{p} + \frac{da-jb}{pb-qa}) \circ (h, 0)$, *for all* p, q *satisfying the above,* d *and* j *are any integers such that* $pd - qj = 1$, *and* $h = \frac{-w \pm 1}{t}$ *where the* $\pm$ *sign agrees with that in (3.2) (note, the choice of* j *and* d *such that* $pd - qj = 1$ *has no effect on* U*).*

Proof. Suppose U is rational. Solving $N(U+\frac{0}{1}) = N(\frac{a}{b})$ gives $U = \frac{a}{b'}$, where $b'b^{\pm 1} = 1 \bmod a$ and $N(\frac{a}{b'} + \frac{t}{w}) = N(\frac{tb'+wa}{ty+wx})$ by lemma 3.

Suppose U is not rational. Since U is a generalized M-tangle which is not rational and $N(a/b)$ is a 4-plat and not a Montesinos knot or link, $U = (U_1 + U_2) \circ (h, 0)$ where U_i are rational but non-integral. If $U_1 = \frac{j}{p}$ where p can be chosen to be positive and d and q are any pair of integers such that $pd - qj = 1$, then solving $N((\frac{j}{p} + U_2) \circ (h,0) + \frac{0}{1}) = N(\frac{j}{p} + U_2) = N(\frac{a}{b})$ gives $U_2 = \frac{da - jb'}{pb' - qa}$ where $b'b^{\pm 1} = 1$ mod a by lemma 4. Note $\frac{j}{p} + \frac{da-jb}{pb-qa} = \frac{j+pi}{p} + \frac{da-jb-(pb-qa)i}{pb-qa} = \frac{j+pi}{p} + \frac{(d+qi)a-(j+pi)b}{pb-qa}$ and $p(d + qi) - q(j + pi) = 1$ if and only if $pd - qj = 1$. Hence if p and q are specified, then the choice of j and d such that $pd - qj = 1$ has no effect on U.

If $b'b^{-1} \cong 1$ mod a, then $b' = b + ak$ for some integer k. The tangle U $= (\frac{j}{p} + \frac{da-j(b+ka)}{p(b+ka)-qa}) \circ (h, 0) = (\frac{j}{p} + \frac{(d-jk)a-jb}{pb-(q-pk)a}) \circ (h, 0) = (\frac{j}{p} + \frac{Da-jb}{pb-Qa}) \circ (h, 0)$ where $Q = q - pk$, $D = d - jk$ and $pD - Qj = p(d - jk) - (q - pk)j = 1$. Thus, the case $b'b^{-1} \cong 1$ mod a is redundant. Suppose $b'b \cong 1$ mod a. Then there exists y such that $b'b - ay = 1$. Let $P = pb' - qa, J = da - jb'$, $Q = py - qb, D = db - jy$. Then $PD - QJ = 1$, $Da - Jb = j$, and $Pb - Qa = p$. Hence $U = (\frac{j}{p} + \frac{da-jb'}{pb'-qa}) \circ (h, 0) = (\frac{Da-Jb}{Pb-Qa} + \frac{J}{P}) \circ (h, 0)$. Therefore, we need only consider $U = (\frac{j}{p} + \frac{da-jb}{pb-qa} +) \circ (h, 0)$ and $U = (\frac{da-jb}{pb-qa} + \frac{j}{p}) \circ (h, 0)$.

$N((U_1 + U_2) \circ (h, 0) + \frac{t}{w}) = N((U_1 + U_2) + (\frac{t}{w}) \circ (h, 0)) = N((U_1 + U_2) + \frac{t}{ht+w})$. Since U_i are non-integral, $\frac{t}{ht+w}$ must be integral since $N(\frac{z}{v})$ is a 4-plat. Hence, $ht + w = \pm 1$ and $N(U_1 + U_2 + \frac{t}{ht+w}) = N(U_1 + U_2 \pm t) = N(U_2 \pm t + U_1) =$ $N(U_2 + U_1 + \pm t)$. Thus $N((\frac{j}{p} + \frac{da-jb}{pb-qa}) \circ (h, 0) + \frac{t}{w}) = N((\frac{da-jb}{pb-qa} + \frac{j}{p}) \circ (h, 0) + \frac{t}{w})$.

$N((\frac{j}{p} + \frac{da-jb}{pb-qa}) \circ (h, 0) + \frac{t}{w}) = N(\frac{j}{p} + \frac{da-jb}{pb-qa} \pm t) = N(\frac{j}{p} + \frac{da-jb\pm(pb-qa)t}{pb-qa}) =$ $N(\frac{j(pb-qa)+p[da-jb\pm(pb-qa)t]}{d(pb-qa)+q[da-jb\pm(pb-qa)t]}) = N(\frac{tp(pb-qa)\pm a}{tq(pb-qa)\pm b})$.

Note that if $w \not\cong \pm 1$ mod t, then U must be rational since $N(\frac{a}{b})$ and $N(\frac{z}{v})$ are 4-plats. Thus (a) holds. For $w \cong \pm 1$ mod t, suppose $p = 1, q = k, j = 0, d = 1$. Then $U = (\frac{j}{p} + \frac{da-jb}{pb-qa} +) \circ (h, 0) = (\frac{0}{1} + \frac{a}{b-ak}) \circ (h, 0) = \frac{a}{b+a(h-k)}$. If $p = b'$, $j = a, q = y, d = b$ where $b'b - ay = 1$, then $U = (\frac{j}{p} + \frac{da-jb}{pb-qa}) \circ (0, h, 0) = (\frac{a}{b'} + \frac{ba-ab}{b'b-ay}) \circ (h, 0) = (\frac{a}{b'} + \frac{0}{1}) \circ (h, 0) = \frac{a}{b'+ah}$. Thus, the formulas given in (b) include the U rational case □.

Corollary 1. *If* $|t| > 1$ *or if* $N(\frac{a}{b}) = N(\frac{1}{0})$, $N(\frac{z}{v}) = N(\frac{2k}{1})$, *then theorem 3 gives all solutions to the system of tangle equations:* $N(U + \frac{0}{1}) = N(\frac{a}{b})$ *and* $N(U + \frac{t}{w}) =$ $N(\frac{z}{v})$ *where* $N(\frac{a}{b})$ *and* $N(\frac{z}{v})$ *are unoriented 4-plats.*

Proof. Theorem 3 gives all solutions to these equations when U is a generalized M-tangle. By theorems 1, 2, U is a generalized M-tangle when $|t| > 1$ or $N(\frac{a}{b}) =$ $N(\frac{1}{0})$, $N(\frac{z}{v}) = N(\frac{2k}{1})$

Example 1: Theorem 3 tells us that if $N(\frac{z}{v})$ can be obtained from $N(\frac{1}{0})$, the unknot, via a (0,+2) move, then $N(\frac{z}{v}) = N(\frac{2pq\mp1}{2q^2})$ where $(p, q) = 1$. Since +2 moves are equivalent to crossing changes (see sections 5, 6), theorem 3 is a generalization of [13, 14, 5, 18], and a similar unpublished theorem of J. Berge. Theorem 3 is also an extension of [6].

Corollary 2. *Suppose* $bx - ay = 1$, $N(U + \frac{0}{1}) = N(\frac{a}{b})$ *and* $N(U + \frac{t}{w}) = N(\frac{z}{v})$ *where* $N(\frac{a}{b})$ *and* $N(\frac{z}{v})$ *are unoriented 4-plats. If* $w \not\cong \pm 1$ *or if* U *is rational, then* $\frac{t}{w} = \frac{xz - av'}{bv' - yz - kt}$ *and* $U = \frac{a}{b+ka}$ *OR* $\frac{t}{w} = \frac{bz - av'}{xv' - yz - kt}$ *and* $U = \frac{a}{x+ka}$ *where* v' *is any integer such that* $v'v^{\pm 1} = 1$ *mod* z. *If* $w \cong \pm 1$ *mod* t, *then* t *divides* $z \mp a$.

Example 2: A program to solve the system of equations in theorem 3 is available at http://www.math.uiowa.edu/∼idarcy/PROG/comput.html. To illustrate the computations needed, the following system of equations will be solved:

$$N(U + \frac{0}{1}) = N(\frac{2}{1}), \quad N(U + \frac{t}{w}) = N(\frac{11}{7}) \tag{3.3}$$

In this case, $a = 2, b = 1, z = 11$, and $v = 7$. Any x and y can be chosen such that $bx - ay = 1$. Let $x = 1$ and $y = 0$. If U is rational, then $\frac{t}{w} = \frac{xz - av'}{bv' - yz - kt}$ or $\frac{bz - av'}{xv' - yz - kt}$ where v' is any integer such that $v'v^{\pm 1} = 1$ mod z. Hence, $v' = 7 + 11i$ or $8 + 11i$ for some integer i. Therefore, if $w \not\cong \pm 1$ mod t, $\frac{t}{w} = \frac{11 - 2(7+11i)}{7 + 11i - kt}$ or $\frac{11 - 2(8+11i)}{8 + 11i - kt}$ and $U = \frac{2}{1+2k}$ for some integers i and k.

For $w \cong \pm 1$ mod t, first solve the equation, $|tp(pb - qa) \pm a| = |z|$, i.e., $tp(p - 2q) \pm 2 = z'$ where $z' = 11$ or -11. Note that both t and p must divide $z \mp a = 11 \mp 2$, and p can be chosen to be positive. Since $\frac{t}{w} = \frac{-t}{-w}$, t can also be chosen to be positive. Therefore we only need to check $t = 1,\ 3,\ 9,\ 13$. For $t = 1$, $(p, q) = (1, -4), (1, -6), (1, 5), (1, 7), (3, 0), (3, 3), (9, 4), (9, 5), (13, 6)$, or $(13, 7)$. For $t = 3$, $(p, q) = (1, -1), (1, 2), (3, 1)$, or $(3, 2)$. For $t = 9, (p, q) = (1, 0)$ or $(1, 1)$. For $t = 13, (p, q) = (1, 0)$ or $(1, 1)$.

For $t = 1, (p, q) = (3, 0), (3, 3)$ are not possible solutions since p and q must be relatively prime. Let $v' = (z'/z)[tq(pb - qa) \pm b] = (z'/z)[tq(p - 2q) \pm 1]$ (note: the $\pm$ signs must agree for z' and v'). Checking the other solutions into the equation $v'v^{\pm 1} = 1$ mod z, leaves t = 3, (p, q) = (1, 2) or (3, 2) as possible solutions to the equation $N(\frac{tp(p-2q) \pm 2}{tq(p-2q) \pm 1}) = N(\frac{11}{7})$. The tangle $\frac{t}{w} = \frac{3}{-3h-1}$ and $U_1 = \frac{j}{p}$, $U_2 = \frac{da - jb}{pb - qa}$ where we can choose any j and d such that $pd - qj = 1$. For (p, q) = (1, 2), let $j = 0$, and $d = 1$. Thus, $U = (\frac{0}{1} + \frac{2}{-3}) \circ (h, 0) = (\frac{2}{-3} + \frac{0}{1}) \circ (h, 0) = \frac{2}{2h-3}$. For $(p, q) = (3, 2)$, let $j = 1$, and $d = 1$. Thus, $U = (\frac{1}{3} + \frac{1}{-1}) \circ (h, 0) = (\frac{1}{-1} + \frac{1}{3}) \circ (h, 0) = \frac{2}{2h-3}$.

Therefore, if $R = \frac{t}{w}$, and $|t| > 1$, then by theorems 1 and 3, the following are all solutions for U and R to the system of equations in (3.3):

$$N(\tfrac{2}{1+2k} + \tfrac{0}{1}) = N(\tfrac{2}{1}), \qquad N(\tfrac{2}{1+2k} + \tfrac{11-2(7+11i)}{7+11i-kt}) = N(\tfrac{11}{7}),\ i, k \in \mathbf{Z}$$

$$N(\tfrac{2}{1+2k} + \tfrac{0}{1}) = N(\tfrac{2}{1}), \qquad N(\tfrac{2}{1+2k} + \tfrac{11-2(8+11i)}{8+11i-kt}) = N(\tfrac{11}{7}),\ i, k \in \mathbf{Z}$$

where $i = 0, k = h - 2$ in the first equation corresponds to the case when $w \cong \pm 1$ mod t.

Although $\frac{t}{w} = \frac{1}{j}$ is not a possible solution to the system of equations in (3.3) if U is a generalized M-tangle, there is no theorem that restricts U to be a generalized M-tangle when $|t| \leq 1$. Consequently, theorem 3 gives no information as to whether or not it is possible to change $N(2/1)$ into $N(11/7)$ via a $(\frac{0}{1}, \frac{1}{j})$ move. In fact such a conversion is possible [4].

Example 3: Techniques similar to those used in example 2 can also be used to solve for U when $\frac{t}{w}$ is known. For example, solving for the tangle U in the system of equations, $N(U+\frac{0}{1}) = N(\frac{5}{-1})$, $N(U+\frac{2}{1}) = N(\frac{7}{-3})$ results in exactly two solutions for U: $U = (\frac{1}{2}+\frac{1}{3}) \circ (-1,0)$ and $U = (\frac{1}{3}+\frac{1}{2}) \circ (-1,0)$.

4. Some equivalent $(0, \frac{t}{w})$ moves

Note that if there is a solution for the unknowns U and R given the system of equations $N(U+\frac{0}{1}) = K_1$ and $N(U+R) = K_2$, that solution will not be unique. For example as shown in Fig. 8, if $N(U+\frac{0}{1}) = K_1$ and $N(U+\frac{t}{w}) = K_2$, then $N([U \circ (h,0)] + \frac{0}{1}) = K_1$ and $N([U \circ (h,0)] + \frac{t}{w-ht}) = K_2$.

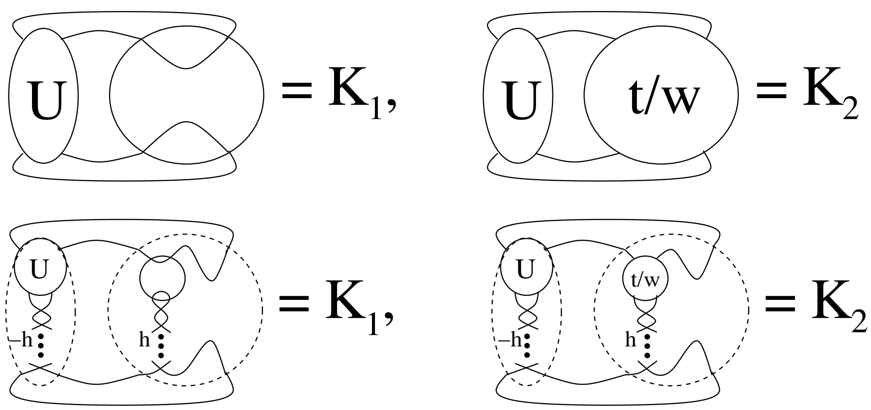

Fig. 8. $N(U+\frac{0}{1}) = N([U \circ (h,0)] + \frac{0}{1}) = K_1$, $N(U+\frac{t}{w}) = N([U \circ (h,0)] + \frac{t}{w-ht}) = K_2$.

Recall that if there exists a solution for U such that $N(U+P) = K_1$ and $N(U+R) = K_2$, then K_2 is said to have been obtained from K_1 by a (P,R) move. Also, a (P,R) move is said to be equivalent to a (P',R') move if there exists a solution for U such the $N(U+P) = K_1$ and $N(U+R) = K_2$ if and only if there exists a solution for U' such the $N(U'+P') = K_1$ and $N(U'+R') = K_2$ for all knots/links K_1, K_2.

Since K_2 can be obtained from K_1 via a $(0, \frac{t}{w})$ move if and only if K_2 can be obtained from K_1 via a $(0, \frac{t}{w-ht})$ move, a $(0, \frac{t}{w})$ move is equivalent to a $(0, \frac{t}{w-ht})$ move. Moreover, these are the only moves that are equivalent when $P = 0$.

Theorem 4. *A $(0, \frac{t}{w})$ move is equivalent to a $(0, \frac{c}{d})$ move if and only if $\frac{c}{d} = \frac{t}{w-ht}$ for some h.*

Proof. By Fig. 8, A $(0, \frac{t}{w})$ move is equivalent to a $(0, \frac{c}{d})$ move if $\frac{c}{d} = \frac{t}{w-ht}$ for some h. Theorem 4 clearly holds if t or c are equal to zero. Since $\frac{c}{d} = \frac{-c}{-d}$, c and similarly t will be taken to be positive. $N(\frac{0}{1}+\frac{0}{1}) = N(\frac{0}{1})$ and $N(\frac{0}{1}+\frac{t}{w}) = N(\frac{t}{w})$. Suppose there exists a U such that $N(U+\frac{0}{1}) = N(\frac{0}{1})$ and $N(U+\frac{c}{d}) = N(\frac{t}{w})$. If $d \cong \pm 1 \bmod c$ and $c \neq 1$, $N(\frac{t}{w}) = N(\frac{cp^2}{cqp \pm 1})$. Thus c divides t. Similarly t divides c. Hence $c = t$. If $c = t = 1$, there exists an h such that $d = w - ht$. Else, $w' \cong cqp \pm 1$

mod t where $w'w^{\pm 1} = 1$ mod t. Therefore, $w \cong \pm 1 \cong d$ mod t. Thus, there exists an h such that $d = w - ht$.

If $d \not\cong \pm 1$ mod c, then $\frac{c}{d} = \frac{t}{w'-kc}$ where $w'w^{\pm 1} = 1$ mod t by corollary 2. Hence $c = t$. Choose $n > 1$ such that $(n, 2t) = 1$. $N(\frac{n}{1} + \frac{0}{1}) = N(\frac{n}{1})$ and $N(\frac{n}{1} + \frac{t}{w}) = N(\frac{nw+t}{w})$. Suppose there exists a U' such that $N(U' + \frac{0}{1}) = N(\frac{n}{1})$ and $N(U' + \frac{c}{d}) = N(\frac{nw+t}{w})$. Hence $\frac{t}{d} = \frac{c}{d} = \frac{nw+t-nw_1}{w_1-ht}$ where $w_1 w^{\pm 1} = 1$ mod $(nw+t)$ by corollary 2. Suppose $nw + t - nw_1 = -t$. Then $n(w - w_1) = -2t$. But this contradicts $n > 1$, $(n, 2t) = 1$. Thus, $nw + t - nw_1 = t$ and $w = w_1$. Hence $d = w_1 - ht = w - ht$ □.

In the next section which (P, R) moves are equivalent to $(0, \frac{t}{w})$ moves will be discussed, where P and R are rational tangles.

5. Solving $N(U + \frac{f_1}{g_1}) = N(\frac{a}{b}), N(U + \frac{f_2}{g_2}) = N(\frac{z}{v})$

Theorem 5 relates $(\frac{f_1}{g_1}, \frac{f_2}{g_2})$ moves to $(\frac{0}{1}, \frac{t}{w})$ moves. Theorem 6 summarizes solving the system of unoriented tangle equations $N(U + \frac{f_1}{g_1}) = N(\frac{a}{b})$ and $N(U + \frac{f_2}{g_2}) = N(\frac{z}{v})$. The following three lemmas will be useful for these calculations.

Lemma 5. *$N(A + (c_1, ..., c_n)) = N(A \circ (c_n, ..., c_1))$, for n odd.*

Proof. Induction on n noting that a rational tangle is invariant under a rotation of $180°$ about the x and y-axes [10] □.

Lemma 6. *$(d_1, ..., d_m) \circ (c_1, ..., c_n) = (d_1, ..., d_m + c_1, ..., c_n)$, when n is odd.*

Note that even if $(c_1, ..., c_n) = (e_1, ..., e_k)$, it is possible that $\frac{a}{b} \circ (c_1, ..., c_n) \neq \frac{a}{b} \circ (e_1, ..., e_k)$. For example, $(1) = (0, 1, 1)$, but $(3) = (2) \circ (1) \neq (2) \circ (0, 1, 1) = (2, 1, 1)$.

The Euler bracket function is needed for the next lemma. Let $E[x_1, ..., x_n]$ be the Euler bracket function which equals the sum of products of the x_i's where zero or more disjoint pairs of consecutive x_i's are omitted [16]. If $n = 0$ then $E[x_1, ..., x_n] = E[] = 1$. If $n < 0$ define $E[x_1, ..., x_n] = 0$. Let $[x_n, ..., x_1]$ denote the continued fraction $x_n + \frac{1}{x_{n-1} + \frac{1}{\dots + \frac{1}{x_1}}}$, the fraction corresponding to the tangle $(x_1, ..., x_n)$. The following useful facts for $n \geq 1$ can be found in [16]:

1.) $E[x_1, ..., x_n] = x_1 E[x_2, ..., x_n] + E[x_3, ..., x_n]$.

2.) $[x_n, ..., x_1] = E[x_1, ..., x_n]/E[x_1, ..., x_{n-1}]$.

3.) Let $a = E[x_1, ..., x_n], b = E[x_1, ..., x_{n-1}]$. If $y = (-1)^{n+1} E[x_2, ..., x_{n-1}]$ and $x = (-1)^{n+1} E[x_2, ..., x_n]$, then $bx - ay = 1$.

Lemma 7. $[c_1, ..., c_n + d_m, ..., d_1] = \frac{E[c_1,...,c_n]E[d_1,...d_{m-1}] + E[c_1,...,c_{n-1}]E[d_1,...d_m]}{E[c_2,...,c_n]E[d_1,...d_{m-1}] + E[c_2,...,c_{n-1}]E[d_1,...d_m]}$.

Proof. Induction on m. See Roberts [16] □.

Theorem 5. *Suppose $f_1/g_1 = (c_1, ..., c_n)$, n odd, where $f_1 = E[c_1, ..., c_n]$ and $g_1 = E[c_1, ..., c_{n-1}]$. Let $e_1 = E[c_2, ..., c_n], i_1 = E[c_2, ..., c_{n-1}]$. If $\frac{f_2}{g_2} = (d_1, ..., d_m)$, let $\frac{t}{w} = \frac{g_1 f_2 - g_2 f_1}{e_1 g_2 - i_1 f_2} = (d_1, ..., d_m - c_n, -c_{n-1}, ..., -c_1)$ and $U' = U \circ (c_n, ..., c_1)$ (or equivalently, if $\frac{t}{w} = (b_1, ..., b_k)$, let $\frac{f_2}{g_2} = \frac{te_1 + wf_1}{ti_1 + wg_1} = (b_1, ..., b_k + c_1, ..., c_n)$, and $U = U' \circ (-c_1, ..., -c_n)$), then for any pair of knots K_1, K_2,*

$$N(U+\frac{f_1}{g_1})=K_1 \qquad N(U+\frac{f_2}{g_2})=K_2$$

if and only if

$$N(U'+\frac{0}{1})=K_1 \qquad N(U'+\frac{t}{w})=K_2$$

Hence an $(\frac{f_1}{g_1},\frac{f_2}{g_2})$ *move is equivalent to a* $(0,\frac{g_1f_2-g_2f_1}{e_1g_2-i_1f_2})$ *move, and similarly, a* $(0,\frac{t}{w})$ *move is equivalent to an* $(\frac{f_1}{g_1},\frac{te_1+wf_1}{ti_1+wg_1})$.

Moreover if $(\frac{f_1}{g_1},\frac{f_2}{g_2})$ *is equivalent to* $(0,\frac{t}{w})$ *then there exists* e_1 *and* i_1 *such that* $g_1e_1-f_1i_1=1$ *and* $\frac{t}{w}=\frac{g_1f_2-g_2f_1}{e_1g_2-i_1f_2}$ *(or equivalently,* $\frac{f_2}{g_2}=\frac{te_1+wf_1}{ti_1+wg_1}$*).*

Proof. $N(U+\frac{f_1}{g_1})=N(U+(c_1,...,c_n))=N(U\circ(c_n,...,c_1)+\frac{0}{1})=N(U'+\frac{0}{1})$ by lemma 5. $N(U+\frac{f_2}{g_2})=N(U+(d_1,...,d_m))=N(U+(d_1,...,d_m)\circ(-c_n,...,-c_1)\circ(c_1,...,c_n))=N(U\circ(c_n,...,c_1)+(d_1,...,d_m-c_n,-c_{n-1},...,-c_1)=N(U'+\frac{t}{w})$ by lemmas 6, 2, and 7. $U'=U\circ(c_n,...,c_1)$ if and only if $U=U\circ(c_n,...,c_1)\circ(-c_1,...,-c_n)=U'\circ(-c_1,...,-c_n)$. If $\frac{t}{w}=(d_1,...,d_m-c_n,-c_{n-1},...,-c_1)=(b_1,...,b_k)$, then $\frac{f_2}{g_2}=(d_1,...,d_m)=(d_1,...,d_m-c_n,-c_{n-1},...,-c_1+c_1,...,c_n)=(b_1,...,b_k+c_1,...,c_n)=\frac{te_1+wf_1}{ti_1+wg_1}$ by lemma 7.

Suppose $(\frac{f_1}{g_1},\frac{f_2}{g_2})$ is equivalent to $(0,\frac{t}{w})$. $(\frac{f_1}{g_1},\frac{f_2}{g_2})$ is equivalent to $(0,\frac{g_1f_2-g_2f_1}{e_1g_2-i_1f_2})$ for any e_1 and i_1 such that $g_1e_1-f_1i_1=1$. Hence $(0,\frac{g_1f_2-g_2f_1}{e_1g_2-i_1f_2})$ is equivalent to $(0,\frac{t}{w})$. Thus $t=g_1f_2-g_2f_1$ and there exists an h such that $e_1g_2-i_1f_2=w-ht$. Thus $w=e_1g_2-i_1f_2+ht=e_1g_2-i_1f_2+h(g_1f_2-g_2f_1)=(e_1-hf_1)g_2-(i_1-hg_1)f_2$ where $g_1(e_1-hf_1)-(i_1-hg_1)f_1=1$ □.

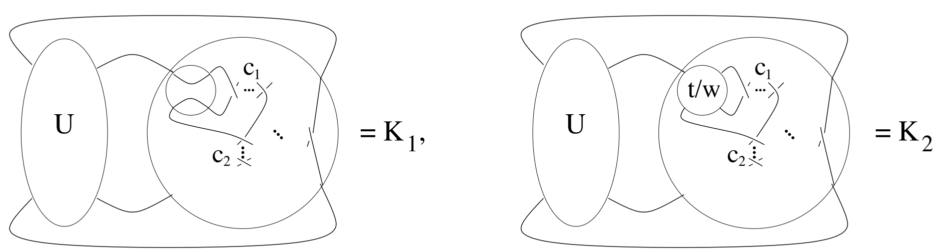

Fig. 9. $N(U+\frac{f_1}{g_1})=K_1$, $N(U+\frac{f_2}{g_2})=K_2$.

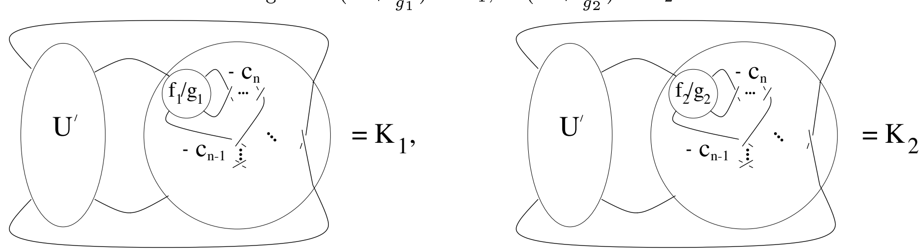

Fig. 10. $N(U'+\frac{0}{1})=K_1$, $N(U'+\frac{t}{w})=K_2$.

Example: In order to solve the system of tangle equations: $N(U + -\frac{1}{3}) = N(\frac{a}{b})$ and $N(U + -\frac{4}{3}) = N(\frac{z}{v})$, note that $-\frac{1}{3} = (-1, -2, 0)$. Thus, $f_1 = E[-1, -2, 0] = -1$, $g_1 = E[-1, -2] = 3$, $e_1 = E[-2, 0] = 1$, $i_1 = E[-2] = -2$ and $\frac{t}{w} = -\frac{4}{3} \circ (0, 2, 1) = \frac{g_1 f_2 - g_2 f_1}{e_1 g_2 - i_1 f_2} = \frac{9}{5}$. Thus, the $(-\frac{1}{3}, -\frac{4}{3})$ move is equivalent to a $(\frac{0}{1}, \frac{9}{5})$ move by theorem 5. Using theorem 3 to solve $N(U' + \frac{0}{1}) = N(\frac{a}{b})$ and $N(U' + \frac{9}{5}) = N(\frac{z}{v})$, we get $N(\frac{z}{v}) = N(\frac{9b'+5a}{9y+5x})$ where $b'b^{\pm 1} = 1$ mod a and x and y are integers such that $b'x - ay = 1$ and $U' = \frac{a}{b'}$.

Thus, by Theorem 5 if $N(U + -\frac{1}{3}) = N(\frac{a}{b})$ and $N(U + -\frac{4}{3}) = N(\frac{z}{v})$, then $N(\frac{z}{v}) = N(\frac{9b'+5a}{9y+5x})$ where $b'b^{\pm 1} = 1$ mod a and x and y are integers such that $b'x - ay = 1$ and $U = \frac{a}{b'} \circ (1, 2, 0) = \frac{E[0,2,1]b' + E[0,2]a}{E[2,1]b' + E[2]a} = \frac{b'+a}{3b'+2a}$ (see also Figs. 11 - 13).

Any continued fraction expansion of $-\frac{1}{3}$ can be used to find a $(0, \frac{t}{w})$ move equivalent to a $(-\frac{1}{3}, -\frac{4}{3})$ move. For example, $-\frac{1}{3} = (-1, 1, k, -3, 0)$. In this case $f_1 = E[-1, 1, k, -3, 0] = -1$, $g_1 = E[-1, 1, k, -3] = 3$, $e_1 = E[1, k, -3, 0] = k + 1$, $i_1 = E[1, k, -3] = -3k - 2$ and $\frac{t}{w} = -\frac{4}{3} \circ (-1, 1, k, -3, 0) = \frac{g_1 f_2 - g_2 f_1}{e_1 g_2 - i_1 f_2} = \frac{9}{9k+5}$. Thus, the $(-\frac{1}{3}, -\frac{4}{3})$ move is also equivalent to a $(\frac{0}{1}, \frac{9}{9k+5})$ move by theorem 5 (or by noting a $(\frac{0}{1}, \frac{9}{5})$ move is equivalent to a $(\frac{0}{1}, \frac{9}{9k+5})$ move).

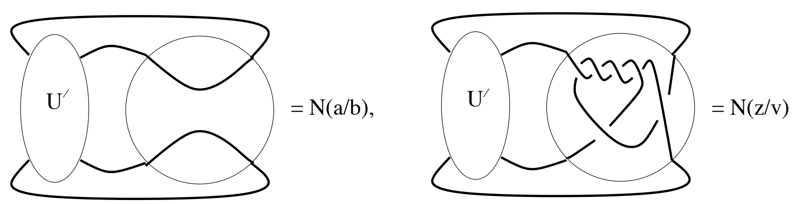

Fig. 11. $N(U' + \frac{0}{1}) = N(\frac{a}{b})$, $N(U' + \frac{9}{5}) = N(\frac{z}{v})$.

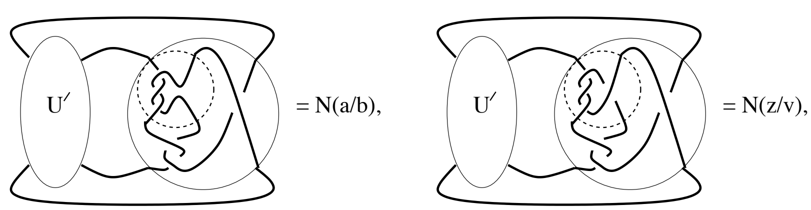

Fig. 12. $N(U' + \frac{0}{1}) = N(\frac{a}{b})$, $N(U' + \frac{9}{5}) = N(\frac{z}{v})$.

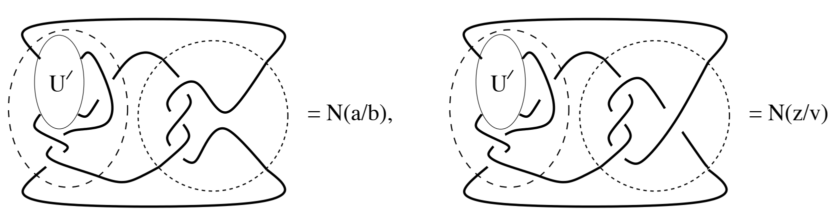

Fig. 13. $N(U' \circ (1, 2, 0) + -\frac{1}{3}) = N(\frac{a}{b})$, $N(U' \circ (1, 2, 0) + -\frac{4}{3}) = N(\frac{z}{v})$.

Alternatively, we can use theorem 6 which follows directly from and summarizes theorems 3, 5, 1, and 2. Also note that in theorem 6, all $\pm$ signs except those involving exponents are in agreement.

Theorem 6. *Suppose* $\frac{f_1}{g_1} = (c_1, ..., c_n), n$ *odd,* $f_1 = E[c_1, ..., c_n], g_1 = E[c_1, ..., c_{n-1}]$, $e_1 = E[c_2, ..., c_n], i_1 = E[c_2, ..., c_{n-1}]$, $t = g_1 f_2 - g_2 f_1$, $w = e_1 g_2 - i_1 f_2$. $N(U + \frac{f_1}{g_1}) = N(\frac{a}{b})$ *and* $N(U + \frac{f_2}{g_2}) = N(\frac{z}{v})$ *where* $N(\frac{a}{b})$ *and* $N(\frac{z}{v})$ *are unoriented 4-plats and* U *is a generalized M-tangle if and only if the following hold:*

(a) If $w \ncong \pm 1$ *mod* t*, then there exists an integer,* b' *such that* $b'b^{\pm 1} = 1$ *mod* a*, and for any integers* x *and* y *such that* $b'x - ay = 1$*,*

$$N(\frac{z}{v}) = N(\frac{tb' + wa}{ty + wx}) \tag{5.4}$$

In this case, $U = \frac{a}{b'} \circ (-c_1, ..., -c_n) = \frac{-f_1 b' + e_1 a}{g_1 b' - i_1 a}$ *for all* b' *satisfying the above.*

(b) If $w \cong \pm 1$ *mod* t*, then there exists relatively prime integers,* p *and* q*, where* p *may be chosen to be positive, such that*

$$N(\frac{z}{v}) = N(\frac{tp(pb - qa) \pm a}{tq(pb - qa) \pm b}) \tag{5.5}$$

In this case, $U = (U_1 + U_2) \circ (h, -c_1, ..., -c_n))$ *and* $U = (U_2 + U_1) \circ (h, -c_1, ..., -c_n)$ *where* $U_1 = \frac{j}{p}$*,* $U_2 = \frac{da - jb}{pb - qa}$ *are both solutions for* U*, for all* p, q *satisfying the above, and* d *and* j *are any integers such that* $pd - qj = 1$*, and* $h = \frac{-w \pm 1}{t}$ *(note, the choice of* j *such that* $pd - qj = 1$ *has no effect on* U*).*

If $|f_1 g_2 - f_2 g_1| > 1$ *or if* $|f_1 g_2 - f_2 g_1| = 1$*,* $N(\frac{a}{b}) = N(\frac{1}{0})$ *and* $N(\frac{z}{v}) = N(\frac{2k}{1})$*, then the above list of solutions to the system of equations,* $N(U + \frac{f_1}{g_1}) = N(\frac{a}{b})$ *and* $N(U + \frac{f_2}{g_2}) = N(\frac{z}{v})$*, is complete.*

Acknowledgments

This work was supported by a grant from the Joint DMS/NIGMS Initiative to Support Research in the Area of Mathematical Biology (NIH GM76242).

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