FOCK SPACE AND KAZHDAN-LUSZTIG POLYNOMIALS

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1. Lecture 1: Affine Lie algebras and the Fock representation of $\widehat{gl}_n$.

1.1. **The loop algebra construction.** Let $\mathfrak{g}$ be a complex reductive Lie algebra and let $\mathcal{L}$ denote the algebra of Laurent polynomials in one variable $\mathcal{L} = \mathbb{C}[t, t^{-1}]$. The *loop algebra* over $\mathfrak{g}$ is $\mathcal{L}(\mathfrak{g}) = \mathcal{L} \otimes \mathfrak{g}$, which is a Lie algebra with the bracket

$$[t^r \otimes x, t^s \otimes y]_0 = t^{r+s}[x, y]. \tag{1.1}$$

The elements of the loop algebra may be regarded as regular rational functions on $\mathbb{C}^\times$ with values in $\mathfrak{g}$. If V is a $\mathfrak{g}$-module, then $\mathcal{L}(V) = \mathcal{L} \otimes V$ is an $\mathcal{L}(\mathfrak{g})$-module with the action

$$(t^r \otimes x)(t^s \otimes v) = t^{r+s}xv. \tag{1.2}$$

Define a 2-cocycle on $\mathfrak{g}$ using a non-degenerate symmetric associative bilinear form $(\cdot, \cdot)$ on $\mathfrak{g}$ (namely, the Killing form on the derived algebra of $\mathfrak{g}$, direct sum any non-degenerate symmetric form on the center of $\mathfrak{g}$.) Define $\alpha : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{C}$ by

$$\alpha(x(t), y(t)) = \mathrm{Res}_0((\mathrm{x}'(\mathrm{t}), \mathrm{y}(\mathrm{t}))). \tag{1.3}$$

In particular,

$$\alpha(t^r \otimes x, t^s \otimes y) = \delta_{r+s,0}\, r\, (x, y). \tag{1.4}$$

Then α is a 2-cocycle, namely it is antisymmetric and satisfies

$$\begin{aligned} \alpha([x(t), y(t)], z(t)) + \alpha([y(t), z(t)], x(t)) \\ + \alpha([z(t), x(t)], y(t)) = 0. \end{aligned} \tag{1.5}$$

For showing this, it is helpful to use the bilinear functional on $\mathcal{L}$ given by

$$\varphi(f, g) = \mathrm{Res}_0(\mathrm{f}'\mathrm{g}),$$

which one can show is antisymmetric and satisfies

$$\varphi(fg, h) + \varphi(gh, f) + \varphi(hf, g) = 0. \tag{1.6}$$

Using the 2-cocycle α, one obtains a central extension $\hat{\mathfrak{g}}'$ of $\mathcal{L}(\mathfrak{g})$:

$$0 \longrightarrow \mathbb{C} \longrightarrow \hat{\mathfrak{g}}' \longrightarrow \mathcal{L}(\mathfrak{g}) \longrightarrow 0. \tag{1.7}$$

More explicitly,

$$\hat{\mathfrak{g}}' = \mathcal{L}(\mathfrak{g}) \oplus \mathbb{C}K \tag{1.8}$$

with the Lie bracket

$$\begin{aligned} &[a(t) + \lambda_1 K,\ b(t) + \lambda_2 K] \\ &\qquad = [a(t), b(t)]_0 + \alpha(a(t), b(t))K. \end{aligned} \tag{1.9}$$

Next note that $D : \hat{\mathfrak{g}}' \longrightarrow \hat{\mathfrak{g}}'$ defined by

$$D(a(t) + \lambda K) = ta'(t)) \tag{1.10}$$

is a derivation of $\hat{\mathfrak{g}}'$, so one can make the further extension of $\hat{\mathfrak{g}}'$

$$\hat{\mathfrak{g}} = \mathcal{L}(\mathfrak{g}) \oplus \mathbb{C}K \oplus \mathbb{C}D, \tag{1.11}$$

with the Lie bracket

$$\begin{aligned} &[a(t) + \lambda_1 K + \mu_1 D,\ b(t) + \lambda_2 K + \mu_2 D] \\ &\qquad = ([a(t), b(t)]_0 + \mu_1 t b'(t) - \mu_2 t a'(t)) \\ &\qquad\qquad + \alpha(a(t), b(t))K \end{aligned} \tag{1.12}$$

Note that $[\hat{\mathfrak{g}}, \hat{\mathfrak{g}}] = \hat{\mathfrak{g}}'$.

Define the bilinear form on $\hat{g}$ by

$$\begin{aligned} &(t^r \otimes x + \lambda_1 K + \mu_1 D,\ t^s \otimes y + \lambda_2 K + \mu_2 D) \\ &\qquad = \delta_{r+s,0}(x, y) + \lambda_1 \mu_2 + \lambda_2 \mu_1. \end{aligned} \tag{1.13}$$

Now one can check that this bilinear form non-degenerate, symmetric, and associative.

1.2. **The Fermionic Fock space construction.** At this point let us specialize to $\mathfrak{g} = \mathfrak{gl}_n$ or $\mathfrak{g} = \mathfrak{sl}_n$; in particular, the non-degnerate, symmetric, associative bilinear form on $\mathfrak{g}$ will be taken to be $(x, y) = tr(xy)$. Let V be the vector representation of $\mathfrak{g}$ with basis $\{u_1, \ldots, u_n\}$. Identify $\mathcal{L}(V) = \mathcal{L} \otimes V$ with $\mathbb{C}^\infty$, with basis $\{v_i : i \in \mathbb{Z}\}$, by the correspondence

$$t^k \otimes u_i = v_{-nk+i}. \tag{1.14}$$

Thus for e_{ij} a matrix unit in $\mathfrak{gl}_n$,

$$(t^r \otimes e_{ij}) v_{ns+\mu} = \delta_{j\mu} v_{ns-nr+i}, \tag{1.15}$$

so $t^r \otimes e_{ij}$ acts on $\mathcal{L}(V)$ by

$$\sum_s E_{ns-nr+i,ns+j}, \tag{1.16}$$

where the $E_{\mu\nu}$ are matrix units in $\mathfrak{gl}_\infty$.

The semi-infinite fermionic Fock space based on $\mathcal{L}(V)$ is the subspace of $\bigwedge_0^\infty \mathcal{L}(V)$ spanned by vectors

$$v_I = v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge \cdots \tag{1.17}$$

where

$$i_0 > i_i > i_2 \cdots , \tag{1.18}$$

and where the set I of indices i_j contains all but finitely many non-positive numbers.

The following language gives us the pleasant illusion that we are talking about physics: for any "state" given by a vector v_I, the positive indices in I label "electrons" and the missing non-positive indices label "positrons". Then F is the direct sum over $m \in \mathbb{Z}$ of $F^{(m)}$, where $F^{(m)}$ has a basis of states in which the number of electrons less the nunber of positrons is m. Let

$$\psi^m = v_m \wedge v_{m-1} \wedge \cdots . \tag{1.19}$$

Then $F^{(m)}$ is the span of $v_I = v_{i_0} \wedge v_{I_1} \wedge \cdots$, where $i_j = m - j$ for $j >> 0$.

Since we have an action of the loop algebra $\mathcal{L}(\mathfrak{g})$ on $\mathcal{L}(V)$, we would like to define an action of the loop algebra on F by

$$\pi(t^r \otimes e_{ij})v_I = \sum_s v_{i_0} \wedge \cdots \wedge (t^r \otimes e_{ij})v_{i_s} \wedge \cdots . \tag{1.20}$$

The sum is finite if $r \neq 0$ or if $i \neq j$ because of the finiteness condition on the vectors v_I. However, if $r = 0$ and $i = j$, then one ends up with an infinite sum. The way to correct this is to define

$$\pi(1 \otimes e_{ii})v_I = N(I, [i])v_I, \tag{1.21}$$

where $N(I, [i])$ is number of indices $j \in I$ such that $j > 0$ and $j \equiv i \pmod n$, less then number of indices $j \notin I$ such that $j \leq 0$ and $j \equiv i \pmod n$. In our colorful physics language, we consider the electrons and positrons to have one of n colors, labelled by residue classes of integers modulo n; then $N(I, [i])$ is the number of $[i]$-colored electrons in the state v_I less the number of $[i]$-colored positrons.

Now it is possible to check that π is a projective representation of the loop algebra, namely

$$\pi(a)\pi(b) - \pi(b)\pi(a) - \pi([a, b]_0) = \alpha(a, b)I. \tag{1.22}$$

In fact, the essential fact to check is that

$$\begin{aligned} \pi(t^r \otimes e_{ij})\pi(t^{-r} \otimes e_{ji}) - \pi(t^{-r} \otimes e_{ji})\pi(t^r \otimes e_{ij})& \\ -\pi(1 \otimes (e_{ii} - e_{jj}) = rI.& \end{aligned} \tag{1.23}$$

It follows that π defines a representation of the central extension $\hat{\mathfrak{g}}'$ on F, in which the central element K acts as the identity. This representation clearly leaves each of the spaces $F^{(m)}$ invariant.

The representation of $\hat{\mathfrak{g}}'$ on $F^{(m)}$ can be extended to a representation of $\hat{\mathfrak{g}}$. See subsection 1.4 below for an explicit extension in the case m = 0.

Consider the usual triangular decomposition of $\mathfrak{g}$,

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+, \tag{1.24}$$

where $\mathfrak{h}$ is the Cartan subalgebra, and $\mathfrak{n}^+$ is the direct sum of the roots spaces for positive roots. Then $\hat{\mathfrak{g}}$ also has a triangular decomposition

$$\hat{\mathfrak{g}} = \hat{\mathfrak{n}}^- \oplus \hat{\mathfrak{h}} \oplus \hat{\mathfrak{n}}^+, \tag{1.25}$$

where

$$\hat{\mathfrak{h}} = (1 \otimes \mathfrak{h}) \oplus \mathbb{C}K \oplus \mathbb{C}D, \tag{1.26}$$

$$\hat{\mathfrak{n}}_+ = (1 \otimes \mathfrak{n}^+) \oplus \sum_{r<0} t^r \otimes \mathfrak{g}, \tag{1.27}$$

and

$$\hat{\mathfrak{n}}_- = (1 \otimes \mathfrak{n}^-) \oplus \sum_{r>0} t^r \otimes \mathfrak{g}. \tag{1.28}$$

Now it is easy to see that ψ^m is a highest weight vector, namely

$$\pi(h)\psi^m \in \mathbb{C}\psi^m, \tag{1.29}$$

for $h \in \mathfrak{h}$, and

$$\pi(x)\psi^m = 0 \tag{1.30}$$

for $x \in \hat{\mathfrak{n}}^+$.

One can also check that $F^{(m)}$ is a simple $\widehat{\mathfrak{gl}}_n$ module. As a $\widehat{\mathfrak{sl}}_n$ module, however, it a direct sum of countably many simple highest weight modules; the cyclic submodule generated by ψ^m is a simple highest weight $\widehat{\mathfrak{sl}}_n$ module.

1.3. **Generators and relations.** When $\mathfrak{g}$ is a simple complex Lie algebra of type X_l, then the extended loop algebra $\hat{\mathfrak{g}}$ constructed above is isomorphic to the affine Kac-Moody Lie algebra of type $X_l^{(1)}$. This correspondence is discussed in Kac, *Infinite Dimensional Lie Algebras*, Chapter 7.

Here I only specify the correspondence for $\mathfrak{g} = \mathfrak{sl}_n$. Let $e_i = e_{i\, i+1}$, $f_i = e_{i+1\, i}$, and $h_i = e_{i\, i} - e_{i+1\, i+1}$, for $1 \le i \le n-1$, be the standard generators of $\mathfrak{sl}_n$. Let us also write e_i for $1 \otimes e_i$ and similarly for f_i. Furthermore, let us put $e_0 = t^{-1} \otimes e_{1n}$, $f_0 = t \otimes e_{n1}$, and $h_0 = K - (e_{11} - e_{nn})$.

Then the set $\{e_i, f_i, h_i\}$ generates $\hat{\mathfrak{g}}'$ as a Lie algebra, and using these generators, one can show that $\hat{\mathfrak{g}}'$ is isomorphic to the Lie algebra $\hat{\mathfrak{g}}'(A)$ determined by the Cartan matrix A of type $A_n^{(1)}$.

Recall that the abelian Lie algebra $\hat{\mathfrak{h}}$ has a basis

$$\{h_0, h_1, \dots, h_{n-1}, D\}.$$

Let

$$\{\omega_0, \omega_1, \dots, \omega_{n-1}, \delta\}$$

be the dual basis of $\hat{\mathfrak{h}}^*$.

Then one can check that $\psi^m \in F^{(m)}$ is a weight vector of weight ω_j, where $j \equiv m \pmod{n}$.

1.4. **Parametrization by Young diagrams.** In this subsection, I restrict attention to the the Fock module $\mathcal{F}$. There is a bijection between the basis vectors v_I of $\mathcal{F}$ and Young diagrams (or partitions) $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0$ of arbitrary size. The correspondence is given by

$$\lambda \mapsto I = (\lambda_1, \lambda_2 - 1, \lambda_3 - 2, \dots),$$

where λ has been augmented by appending infinitely many zeros. Thus $\lambda_j - j + 1 = i_{j-1}$.

We now obtain formulas for the action of the generators e_i, f_i, h_i and D of $\widehat{\mathfrak{sl}}_n$ on basis elements of $\mathcal{F}$ labelled by Young diagrams.

The generator f_i, applied to a basis vector v_I, produces new basis vectors by increasing some index i_j, whenever it is possible to increase the index i_j (either $j = 0$ or $i_{j-1} > i_j + 1$) and when moreover $i_j \equiv i \pmod{n}$.

In terms of Young diagrams this translates into adding a box to the the Young diagram in some row i with λ_i boxes, whenever it is possible to add a box (that is, $i = 1$ or $\lambda_{i-1} > \lambda_i$) and when moreover $\lambda_i + 1 - i$ is congruent to i modulo n.

The *content* of a box or node in a Young diagram is its column index minus its row index. The *residue* of a node is the residue modulo n of the content. The diagram λ is said to have a *removable r-node* at (i, λ_i) if the node can be removed to obtain a smaller Young diagram, and if the residue of the node is r. It is said to have a *indent r-node* at $(i, \lambda_i + 1)$ if the node can be added to obtain a larger Young diagram, and if the residue of the node is r.

The prescription for the action of f_i is thus

$$f_i \lambda = \sum_{\nu} \nu, \tag{1.31}$$

where the sum is over all diagrams ν which can be obtained from λ by adding one node of residue i.

Similarly

$$e_i\lambda = \sum_{\nu} \nu, \tag{1.32}$$

where the sum is over all diagrams ν which can be obtained from λ by removing one node of residue i.

The prescription for the action of the h_i is

$$h_i\lambda = n(\lambda, i)\lambda, \tag{1.33}$$

where $n(\lambda, i)$ is the number of indent i-nodes of λ less the number of removable i-nodes.

The formula for the action of D is

$$D\lambda = n_0(\lambda)\lambda, \tag{1.34}$$

where $n_0(\lambda)$ the total number of nodes of λ of residue 0 modulo n. (To see this, note that D commutes with all h_i and with all e_i and f_i for $i \neq 0$. However, $Df_0 = f_0(D+1)$, and $De_0 = e_0(D-1)$. It follows that D counts the number of 0-nodes of a Young diagram.)

The point of reciting these formulas here is to provide a basis for comparison for analogous formulas for the quantum universal enveloping algebra $U_v(\widehat{\mathfrak{sl}}_n)$.

(Note the formula for $h_i\lambda$ follows easily from $h_i = [e_i, f_i]$. Consequently, for $i \neq 0$ one must have

$$n(\lambda, i) = N(I, [i]) - N(I, [i+1]), \tag{1.35}$$

where $N(I, [i])$ is defined after Equation (1.21), and I is the sequence of indices corresponding to the Young diagram λ. For $i = 0$, one has a modified formula:

$$n(\lambda, 0) = 1 + N(I, [0]) - N(I, [1]), \tag{1.36}$$

One can also check these formulas directly.)

References for this lecture are:

1. V.G. Kac, *Infinite Dimensional Lie Algebras.*
2. V.G. Kac and A. K. Raina, *Bombay Lectures On Highest Weight Representations of Infinite Dimensional Lie Algebras* .

2. Lecture 2: The Fock representation of $U_v(\widehat{\mathfrak{sl}}_n)$.

2.1. The quantum universal enveloping algebra $U_v(\widehat{\mathfrak{sl}}_n)$. To describe the quantum universal enveloping algebra $U_v(\widehat{\mathfrak{sl}}_n)$ associated to $\widehat{\mathfrak{sl}}_n$, let us start with the Cartan data for $\widehat{\mathfrak{sl}}_n$: Let $\hat{\mathfrak{h}}$ be the $\mathbb{Q}$-vector space with basis

$$\{h_0, h_1, \ldots, h_{n-1}, D\},$$

and let

$$\{\omega_0, \omega_1, \ldots, \omega_{n-1}, \delta\}$$

be the dual basis of $\hat{\mathfrak{h}}^*$. Also let

$$\{\alpha_0, \alpha_1, \ldots, \alpha_{n-1}\} \subset \hat{\mathfrak{h}}^*$$

be the functionals (simple roots):

$$\begin{aligned} \alpha_i &= 2\omega_i - \omega_{i-1} - \omega_{i+1} \quad \text{for } i \neq 0, n \\ \alpha_n &= 2\omega_n - \omega_0 - \omega_{n-1} \\ \alpha_0 &= 2\omega_0 - \omega_n - \omega_1 - \delta, \end{aligned} \tag{2.1}$$

and $A = [\alpha_i(h_j)]$ be the n-by-n Cartan matrix for $\widehat{\mathfrak{sl}}_n$.

Let us first recall the presentation of the ordinary universal enveloping algebra of $\widehat{\mathfrak{sl}}_n$ by generators and relations: $U(\widehat{\mathfrak{sl}}_n)$ is the associative algebra with identity over $\mathbb{Q}, \mathbb{C}, \ldots$, with generators

$$e_i, f_i, h_i \quad \text{for } 0 \leq i \leq n-1 \text{ and } D,$$

satisfying the (Serre) relations are

$$\begin{aligned} &[h, e_j] = \alpha_j(h) e_j, \quad \text{and} \\ &[h, f_j] = -\alpha_j(h) f_j, \quad \text{for } h \in \{h_0, \ldots, h_{n-1}, D\}; \\ &[e_i, f_j] = \delta_{ij} h_j; \\ &e_i^2 e_{i\pm1} - 2 e_i e_{i\pm1} e_i + e_i e_{i\pm1}^2 = 0, \quad \text{and} \\ &f_i^2 f_{i\pm1} - 2 f_i f_{i\pm1} f_i + f_i f_{i\pm1}^2 = 0; \\ &[e_i, e_j] = [f_i, f_j] = 0 \quad \text{if } i \neq j \pm 1. \end{aligned} \tag{2.2}$$

In these relations it is understood that $n+1$ is to be read as 0 and -1 as n.

The quantum universal enveloping algebra $U_v(\widehat{\mathfrak{sl}}_n)$ is the associative algebra with identity over $\mathbb{Q}(v)$, with generators

$$e_i, f_i \quad \text{for } 0 \le i \le n-1$$

and

$$v^{\pm h} \quad \text{for } h \in \{h_0, \dots, h_{n-1}, D\},$$

satisfying the (quantum Serre) relations:

$$\begin{aligned} &v^h v^{-h} = v^{-h} v^h = 1; \\ &v^h e_j v^{-h} = v^{\alpha_j(h)} e_j, \quad \text{and} \\ &v^h f_j v^{-h} = v^{-\alpha_j(h)} f_j, \quad \text{for } h \in \{h_0, \dots, h_{n-1}, D\}; \\ &[e_i, f_j] = \delta_{ij} \frac{v^{h_i} - v^{-h_i}}{v - v^{-1}}; \\ &e_i^2 e_{i\pm1} - (v + v^{-1}) e_i e_{i\pm1} e_i + e_i e_{i\pm1}^2 = 0, \quad \text{and} \\ &f_i^2 f_{i\pm1} - (v + v^{-1}) f_i f_{i\pm1} f_i + f_i f_{i\pm1}^2 = 0; \\ &[e_i, e_j] = [f_i, f_j] = 0 \quad \text{if } i \ne j \pm 1. \end{aligned} \tag{2.3}$$

The only "surprise" in these relations is the difference quotient appearing in the expression for the commutator of an e and an f. The $v + v^{-1}$ is just the v-integer $[2] = \dfrac{v^2 - v^{-2}}{v - v^{-1}}$; one should expect in general that integers get replaced by v-integers in this theory.

$U_v(\widehat{\mathfrak{sl}}_n)$ is a Hopf algebra; for the moment, I omit the formulas for the comultiplication and antipode.

2.2. **The Fock module $\mathcal{F}$.** I describe a representation of $U_v(\widehat{\mathfrak{sl}}_n)$ on a v-analogue of Fock space $\mathcal{F}$. Let $\mathcal{F}$ be the $\mathbb{Q}(v)$-vector space with a basis of Young diagrams of arbitrary size. One can define an action of $U_v(\widehat{\mathfrak{sl}}_n)$ on $\mathcal{F}$ by the following formulas:

$$f_i \lambda = \sum_{\nu} v^{n^+(\lambda,\nu)} \nu, \tag{2.4}$$

where the sum is over all Young diagrams ν obtained from λ by adding one node of residue i modulo n, and $n^+(\lambda, \nu)$ is defined as follows: if ν is obtained from λ by adding a node of residue i in a certain row r, then $n^+(\lambda, \nu)$ is the number of indent i-nodes in rows $r' < r$ less the number of removable i-nodes in such rows.

$$e_i\lambda = \sum_{\nu} v^{n^-(\lambda,\nu)}\nu, \tag{2.5}$$

where the sum is over all Young diagrams ν obtained from λ by removing one node of residue i modulo n, and $n^-(\lambda, \nu)$ is defined as follows: if ν is obtained from λ by removing a node of residue i in a certain row r, then $n^-(\lambda, \nu)$ is the number of removable i-nodes in rows $r' > r$ less the number of indent i-nodes in such rows.

$$v^{h_i}\lambda = v^{n(\lambda,i)}\lambda, \tag{2.6}$$

where $n(\lambda, i)$ is as before, the number of indent i-nodes of λ less the number of removable i-nodes, and

$$v^D\lambda = v^{n_0(\lambda,i)}\lambda, \tag{2.7}$$

where $n_0(\lambda)$ the total number of nodes of λ of residue 0 modulo n, as before.

In summary, the v^h for $h \in \mathfrak{h}$ act diagonally in the basis of Young diagrams, and the formulas for the eigenvalues are obtained just by exponentiating the old formulas. The f_i act as creation, or raising operators, as before, the e_i as annihilation, or lowering operators, but integer powers of the deformation variable v now enter into the formulas.

The Fock space $\mathcal{F}$ is again a direct sum of simple highest weight modules for $U_v(\widehat{\mathfrak{sl}}_n)$. The cyclic submodule generated by the empty Young diagram is simple with highest weight ω_0,

$$L(\omega_0) \cong U_v(\widehat{\mathfrak{sl}}_n)\emptyset \tag{2.8}$$

$\mathcal{F}$ is a simple module for $U_v(\widehat{\mathfrak{gl}}_n)$, a "slightly" larger quantum universal enveloping algebra.

2.3. **Generalities on crystal bases.** One of the distinctive new features of quantum groups is the appearance of canonical bases of modules, with many remarkable properties. In particular, Kashiwara introduced the notion of "crystal base," which is a basis "at $v = 0$" for which the action of the generators e_i, f_i is especially simple. There are general existence and uniqueness theorems for crystal bases, but it remains a problem in many cases to find explicit combinatorial parametrizations of the crystal base. Here I give the solution of this problem for the simple integrable highest weight module $L(\omega_0)$ for $U_v(\widehat{\mathfrak{sl}}_n)$, for which we have the explicit Fock space model.

First let us introduce the divided powers of the f_i and e_i. Let $[k]$ denote the v-integer

$$[k] = \frac{v^k - v^{-k}}{v - v^{-1}}, \tag{2.9}$$

and $[k]!$ the v-factorial

$$[k]! = [k][k-1]\cdots[2][1]. \tag{2.10}$$

The divided powers of the f_i and e_i are

$$f_i^{(k)} = f_i^k/[k]! \quad e_i^{(k)} = e_i^k/[k]!. \tag{2.11}$$

Next, modified versions $\tilde{e}_i$ and $\tilde{f}_i$ of the e_i and f_i are introduced. For each i, consider the unital subalgebra of $U_v(\widehat{\mathfrak{sl}}_n)$ generated by e_i, f_i, and $v^{\pm h_i}$; this is is isomorphic $U_v(sl_2)$, and is denoted $U_v(sl_2)_i$. Any integrable $U_v(\widehat{\mathfrak{sl}}_n)$ module M is a direct sum of finite dimensional simple $U_v(sl_2)_i$ modules, $M = \bigoplus V_s$, and each V_s has a basis of the form

$$u_0,\ f_i^{(1)}u_0,\ f_i^{(2)}u_0, \dots, f_i^{(m)}u_0,$$

where $e_i u_0 = f_i^{(m+1)}u_0 = 0$. On each V_s, $\tilde{e}_i$ and $\tilde{f}_i$ are defined by

$$\begin{aligned} \tilde{f}_i(f_i^{(k)}u_0) &= f_i^{(k+1)}u_0, \\ \tilde{e}_i(f_i^{(k)}u_0) &= f_i^{(k-1)}u_0. \end{aligned} \tag{2.12}$$

The linear extension of $\tilde{f}_i$ and $\tilde{e}_i$ to M is independent of the decomposition $M = \bigoplus V_s$.

Let $\mathcal{A} \subset \mathbb{Q}(v)$ denote the local ring of rational functions with no pole at $v = 0$. An *admissable* $\mathcal{A}$-lattice $\mathcal{M}$ in an integrable $\widehat{\mathfrak{sl}}$ module M is one which is invariant under all $\tilde{e}_i$ and $\tilde{f}_i$. A *crystal base* of M is a pair $(\mathcal{M}, \mathcal{B})$ consisting of an admissable $\mathcal{A}$-lattice $\mathcal{M}$ and a basis $\mathcal{B}$ of $\mathcal{M}/v\mathcal{M}$ with the crucial property: for all $b \in \mathcal{B}$ and all i,

$$\tilde{f}_i b \in \mathcal{B} \cup \{0\}, \text{ and } \tilde{e}_i b \in \mathcal{B} \cup \{0\}, \tag{2.13}$$

and furthermore,

$$\tilde{f}_i b = b' \iff b = \tilde{e}_i b'. \tag{2.14}$$

(I omit some additional technical conditions.) The *crystal graph* is a colored directed graph whose vertices are the elements of the crystal base and whose directed colored edges

$$b \xrightarrow{i} b' \tag{2.15}$$

correspond to pairs $b, b' \in \mathcal{B}$ such that $\tilde{f}_i b = b'$.

2.4. **The crystal base of $L(\omega_0)$.** As mentioned before, there are general existence and uniqueness results for crystal bases. I now describe an explicit parametrization of the crystal base of the Fock module $\mathcal{F}$ and of the simple integrable highest weight module $L(\omega_0)$ of $U_v(\widehat{\mathfrak{sl}}_n)$.

Let $L(\mathcal{F}) \subset \mathcal{F}$ be the $\mathcal{A}$-span of all Young diagrams, and let

$$\mathcal{B}(\mathcal{F}) = \{\lambda + vL(\mathcal{F}) : \lambda \text{ a Young diagram}\}.$$

Furthermore, put $M = U_v(\widehat{\mathfrak{sl}}_n)\emptyset \subseteq \mathcal{F}$, the cyclic submodule generated by the empty Young diagram, and let $\mathcal{M} = L(\mathcal{F}) \cap M$.

A Young diagram is called *n-regular* if it has no more than $n-1$ rows of any length. (Such diagrams arise in the representation theory of the symmetric group in positive characteristic, the representation theory of Hecke algebras at roots of unity, the representation theory of algebraic groups in positive characteristic.) Let

$$\mathcal{B} = \{\lambda + v\mathcal{M} : \lambda \text{ is } n\text{-regular }\} \tag{2.16}$$

Theorem 2.1. *(Misra and Miwa)*

1. *$(L(\mathcal{F}), \mathcal{B}(\mathcal{F}))$ is a crystal base of $\mathcal{F}$.*
2. *$(\mathcal{M}, \mathcal{B})$ is a crystal base of $M \cong L(\omega_0)$.*

The crystal graphs have a fairly straighforward combinatorial description the details of which I will omit. In brief,

$$(\lambda + v\mathcal{M}) \xrightarrow{i} (\nu + v\mathcal{M}) \tag{2.17}$$

if, and only if, ν is obtained from λ by addition of a node of residue i satisfying an additional combinatorial condition; for given λ, there is at most one such node. The crystal graph $\mathcal{B}$ of M is the connected component of the empty diagram in the crystal graph $\mathcal{B}(\mathcal{F})$ of $\mathcal{F}$.

2.5. **The global crystal base.** For a simple integrable highest weight module M, there is in general a unique lifting of the crystal base (a basis of $\mathcal{M}/v\mathcal{M}$) to a basis of M satisfying a self-duality condition. This lifting is called the *global crystal base*, and has been shown to coincide with the canonical bases defined by Lusztig, using the theory of perverse sheaves (whatever that might be!)

There is an involution $a \mapsto \bar{a}$ of $U_v(\widehat{\mathfrak{sl}}_n)$ such that the e_i and f_i are self-dual, $\bar{v} = v^{-1}$, and $(v^h)^- = v^{-h}$. This induces an involution on M by $(a\emptyset)^- = \bar{a}\emptyset$.

Theorem 2.2. (Kashiwara) Let $(\mathcal{M}, \mathcal{B})$ be a crystal base of the simple integrable highest weight module M. Then there is a unique $\mathcal{A}$-basis $\{G(b) : b \in \mathcal{B}\}$ of $\mathcal{M}$ satisfying:

(G1) $G(b) + v\mathcal{M} = b$; and

(G2) $G(b)$ is self-dual.

In our context, one has

Theorem 2.3. (Kashiwara + Misra-Miwa) There is a unique $\mathcal{A}$-basis $\{G(\mu)\}$ of $L(\mathcal{F}) \cap M$ indexed by n-regular Young diagrams μ, and satisfying:

(G1) $G(\mu) \equiv \mu \pmod{vL(\mathcal{F})}$; and

(G2) $G(\mu)$ is self-dual.

2.6. **Ariki's Theorem.** Consider the expansion of the elements $G(\mu)$ of the global crystal base of

$$M = U_v(\widehat{\mathfrak{sl}}_n)\emptyset \subset \mathcal{F}$$

in terms of the natural basis of $\mathcal{F}$ consisting of Young diagrams:

$$G(\mu) = \sum_{\lambda} d_{\lambda,\mu}(v)\lambda \tag{2.18}$$

One has that $d_{\mu\mu} = 1$, and the remaining coeficients are in $v\mathcal{A}$. Lascoux, Leclerc, and Thibon have given an algorithm for the $G(\mu)$ which shows that in fact the $d_{\lambda,\mu} \in \mathbb{Z}[v]$, and $d_{\lambda,\mu} = 0$ unless $\lambda \trianglelefteq \mu$ in dominance order.

LLT conjectured (on the basis of computations) that $d_{\lambda,\mu}(1)$ are decomposition numbers for the Hecke algebra $H_f(q)$ of type A over a field of characteristic 0, with parameter q a primitive n-th root of unity. This remarkable result was proved by Ariki in 1996.

It was shown moreover by Varagnolo and Vasserot in 1998 that $d_{\lambda,\mu}(v) \in \mathbb{N}[v]$.

It remains here to indicate what is meant by the decomposition numbers for the Hecke algebra. The Hecke algebra $H_f(q)$ over $\mathbb{Q}$ is the associative algebra with a basis

$$\{T_w : w \in \text{ the symmetric group } S_f\}$$

satisfying

$$T_{s_i}T_w = \begin{cases} T_{s_i w} & \text{if } \ell(s_i w) = \ell(w) + 1 \\ (q-1)T_w + T_{s_i w} & \text{if } \ell(s_i w) = \ell(w) - 1 \end{cases} \tag{2.19}$$

When $q = 1$, one obtains the group algebra of the symmetric group. When q is not a proper root of unity, then the Hecke algebra is semisimple, and isomorphic to the group algebra of the symmetric group. However, when q is an n-th root of unity and $f \geq n$, then $H_f(q)$ is not semisimple.

In this case, the simple modules D^μ are labelled by n-regular Young diagrams μ of size f.

On the other hand, for each Young diagram of size f, there is a canonical indecomposible (Specht) module S^λ, which is constructed, roughly speaking, by a process of symmetrization in the rows of λ and antisymmetrization in the columns, as for the construction of the simple S_f modules (in characteristic 0). The Specht modules have the same dimensions as the simple S_f modules.

The Specht module S^λ has a *composition series* with simple subquotients D^μ. The *decomposition number* $d_{\lambda,\mu}$ is the number of times D^μ appears in a composition series for S^λ. It is known that $d_{\mu\mu} = 1$ and $d_{\lambda,\mu} = 0$ unless $\lambda \trianglelefteq \mu$.

Theorem 2.4. (Ariki)

$$d_{\lambda,\mu}(1) = d_{\lambda,\mu}.$$

Let us summarize the development described in this lecture: The basic module $L(\omega_0)$ of the quantum universal enveloping algebra $U_v(\widehat{\mathfrak{sl}}_n)$ has a model based on a Fermionic Fock space construction; the Fermionic Fock space $\mathcal{F}$ has a basis labelled by Young diagrams. One can give an explicit combinatorial parametrization of the crystal base, and global crystal base, of this module, by n-regular Young diagrams. The global crystal base can be expanded in terms of the natural basis of $\mathcal{F}$ consisting of Young diagrams:

$$G(\mu) = \sum_\lambda d_{\lambda,\mu}(v)\lambda, \tag{2.20}$$

and the coefficents $d_{\lambda,\mu}(v)$ lie in $\mathbb{Z}[v]$. This expansion can be computed by a combinatorial algorithm due to Lascoux, Leclerc, and Thibon. Finally, evaluating the coefficients $d_{\lambda,\mu}(v)$ at $v = 1$ gives the decomposition numbers for the Hecke algebras $H_f(q)$, where q is an n-th root of unity.

3. Lecture 3: Tilting modules and Kazhdan-Lusztig polynomials.

I now want to describe some aspects of the represntation theory of quantum groups at roots of unity. Let $\mathfrak{g}$ be a simple complex Lie algebra, and consider the quantum universal enveloping algebra $U_q(\mathfrak{g})$, where the parameter q is a primitive n-th root of unity.

This is an algebra over $\mathbb{C}$ or $\mathbb{Q}$ with generators and relations involving the Cartan data for $\mathfrak{g}$. Actually, there are several possible versions for the quantum group at a root of unity; we need to use one introduced by Lusztig which includes divided powers of the generators e_α, f_α.

The algebras $U_q(\mathfrak{g})$ are Hopf algebras with a non-semisimple finite dimensional representation theory. Advertisement: The representation theory is used to construct invariants of 3-manifolds, topological quantum field theories. A part of the representation theory gives the fusion in WZW conformal field theories.

$U_q(\mathfrak{g})$ has finite dimensional simple modules L_λ labelled by dominant integral weights of $\mathfrak{g}$. It also has *standard* modules ∇_λ which have largest semisimple submodule isomorphic to L_λ and *co-standard* modules Δ_λ with largest semisimple quotient isomorphic to L_λ.

There is a certain category of finite dimensional modules, called *tilting modules*, which has proved very useful.

- Tilting modules are characterized by the existence of filtrations by ∇_λ's and by Δ_μ's.
- There is a unique indecomposable tilting module T_λ with highest weight λ.
- For $\mathfrak{g} = \mathfrak{sl}_k$, the tilting modules are characterized as direct summands of some tensor power of the vector representation.

One has some "decomposition numbers" in this theory, namely

$$(T_\mu : \Delta_\lambda) = (T_\mu : \nabla_\lambda)$$

which are multiplicities of Δ_λ in a Δ-filtration of T_μ, or of ∇_λ in a ∇-filtration.

For $\mathfrak{g} = \mathfrak{sl}_k$ and μ n-regular, it is known that

$$(T_\mu : \nabla_\lambda) = d_{\lambda,\mu} = [S^\lambda : D^\mu],$$

the multiplicity of the simple $H_f(q)$ module D^μ in the Specht module S^λ. This is due to a version of Schur-Weyl duality.

W. Soergel conjectured (and then proved at least for Lie types ADE, and probably for all types) that the multiplicities of standard modules in indecomposable tilting modules are given by certain parabolic, affine Kazhdan-Lusztig polynomials, evaluated at 1,

$$(T_\mu : \nabla_\lambda) = n_{\lambda+\rho,\mu+\rho}(1) \tag{3.1}$$

For now, let me just say that the $n_{\lambda,\mu}(v)$ are polynomials depending on two dominant integral weights λ, μ, which are defined by some recursive combinatorial prescription.

In type A, we have, therefore, two polynomial analogues of decomposition numbers $n_{\lambda,\mu}(v)$ (Kazhdan-Lusztig) and $d_{\lambda,\mu}(v)$ (from the global crystal base), with:

$$n_{\lambda+\rho,\mu+\rho}(1) = (T_\mu : \nabla_\lambda) = [S^\lambda : D^\mu] = d_{\lambda,\mu}(1) \tag{3.2}$$

Wenzl and I have shown (by elementary combinatorics) that the two polynomial analogues coincide:

Theorem 3.1. (Goodman-Wenzl, Varagnolo-Vasserot)) *For type A,*

$$d_{\lambda,\mu}(v) = n_{\lambda+\rho,\mu+\rho}(v)$$

for μ n-regular and λ arbitrary.

We have, furthermore, given a *fast algorithm*, for all Lie types, for computing the Kazhdan Lusztig polynomials $n_{\lambda,\mu}$. This algorithm is based on the algorithm of Lascoux, Leclerc and Thibon for the $d_{\lambda,\mu}(v)$, generalized to all Lie types. In type A, it is vastly more efficient than the original LLT algorithm. For all Lie types, it is vastly more efficient than the original recursive algorithm for the parabolic, affine Kazhadan-Lusztig polynomials.

Here are some timing figures in type A (with a 266 mhz G-whiz G3 Macintosh): The k in the table corresponds to $\mathfrak{sl}_k$, the n to the order of the root of unity.

$k,\ n,\ \ \mu$	LLT, secs.	GW, secs.	Soergel, secs.
$4, 5,\ [4l, 2l, 0, 0]$	13.5	1.43	5.22
$4, 5,\ [8l, 4l, 0, 0]$	2013	5.95	179
$5, 6,\ [6l, 4l, 2l, 0, 0]$	56924	8.52	6771
$5, 6,\ [12l, 8l, 4l, 0, 0]$	> 3 days	50	> 3 days

3.1. **Parablolic affine Kazhdan Lusztig polynomials.** To describe the KL polynomials, I need some usual paraphanalia of Lie theory, associated with the simple Lie algebra $\mathfrak{g}$:

- R is the root lattice, P the weight lattice.
- $E = R \otimes_{\mathbb{Z}} \mathbb{R}$
- $\mathcal{C}$ is the Weyl chamber, and $\mathcal{C}^0$ its interior.
- W_f is the Weyl group
- $\mathcal{W} = W_f \times nR$, semidirect product of the Weyl group and n times the root lattice, the affine Weyl group, which acts on E.

- An *alcove* is a connected component of the complement in E of the union of affine reflection hyperplanes for W. The set of all alcoves is designated by $\mathcal{A}$, and the set contained in the positive Weyl chamber by $\mathcal{A}^+$.
- A_0 is that unique alcove in $\mathcal{A}^+$ which contains the origin 0 in its closure.
- $\mathcal{S}$ is the set of reflections in the walls of A_0. Then $(\mathcal{W}, \mathcal{S})$ is a Coxeter group.
- As for any Coxeter group, there is a Hecke algebra associated with $(\mathcal{W}, \mathcal{S})$, denoted $\mathcal{H}$. It is the associateive $\mathbb{Z}[v, v^{-1}]$ algebra with identity and a basis $\{H_w : w \in \mathcal{W}\}$ satisfying:

$$H_w H_s = \begin{cases} H_{ws} & \text{if } \ell(ws) = \ell(w) + 1 \\ (v^{-1} - v)H_w + H_{ws} & \text{if } \ell(ws) = \ell(w) - 1 \end{cases} \tag{3.3}$$

  for $w \in \mathcal{W}$ and $s \in \mathcal{S} \subseteq \mathcal{W}$.

The affine Weyl group acts freely and transitively on the set $\mathcal{A}$ of alcoves (on the left) and also on the right by

$$(wA_0)s = wsA_0.$$

The Hecke algebra $\mathcal{H}$ has an involution determined by $v \mapsto v^{-1}$ and $H_x \mapsto (H_{x^{-1}})^{-1}$ The elements

$$C_s = H_s + v \quad s \in \mathcal{S}$$

are self-dual generators of $\mathcal{H}$.

Let $\mathcal{N} = \mathbb{Z}[v, v^{-1}]\mathcal{A}^+$. Then $\mathcal{H}$ acts on $\mathcal{N}$ (on the right) by

$$AC_s = \begin{cases} As + vA & \text{if } As \in \mathcal{A}^+ \text{ and } As \succ A; \\ As + v^{-1}x & \text{if } As \in \mathcal{A}^+ \text{ and } As \prec A; \,, \\ 0 & \text{if } As \notin \mathcal{A}^+, \end{cases} \tag{3.4}$$

where now the inequalities have a geometric interpretation: $As \succ A$ if As is on the positive side of the hyperplane separating the two alcoves.

Proposition 3.2. *There is a unique $\mathbb{Z}[v, v^{-1}]$ basis $\{\bar{N}_A : A \in \mathcal{A}^+\}$ of $\mathcal{N}$ such that*

1. *$\bar{N}_A$ is self-dual.*
2. *$\bar{N}_A = \sum_{B \preceq A} n_{B,A}(v)B$,*

where $n_{A,A} = 1$, and $n_{B,A}(v) \in v\mathbb{Z}[v]$ if $B \neq A$.

The $n_{B,A}(v)$ are the parabolic affine Kazhdan Lusztig polynomials. (Note that, for the moment, they are parametrized by pairs of alcoves rather than pairs of dominant integral weights.)

The $\bar{N}_A$ can be computed by a recursive scheme (this is the existence proof). One has $\bar{N}_{A_0} = N_{A_0}$. Given $A \neq A_0$, one can choose $s \in \mathcal{S}$ such that $As \in \mathcal{A}^+$ and $As \prec A$. As a first approximation to $\bar{N}_A$ one takes

$$\bar{N}_{As} C_s = N_A + \sum_{B \prec A} f_{B,A}(v) N_B.$$

This element is self-dual, but may have coefficients with non-zero constant term. So one corrects these coefficients by subtracting a self-dual linear combination of $\bar{N}_B$ for $B \prec A$.

Now consider dominant integral weights μ, λ such that $\lambda \trianglelefteq \mu$ and λ is in the $\mathcal{W}$-orbit $\mathcal{W}\mu$. Define $a^+(\lambda)$ to be the unique alcove A such that λ is in the closure of A, and A lies on the positive side of any hyperplane containing λ. (In particular if λ is in the interior of an alcove A, then $a^+(\lambda) = A$.) Then we define

$$n_{\lambda,\mu}(v) = n_{a^+(\lambda), a^+(\mu)},$$

and

$$\bar{N}_\mu = \sum_\lambda n_{\lambda,\mu}(v) \lambda.$$

These are the Kazhdan-Lusztig polynomials indexed by pairs of dominant integral weights which enter into Soergel's theorem, and the result of Goodman-Wenzl mentioned before.

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